SEQUENTIAL TESTS OF THE HYPERGEOMETRIC DISTRIBUTION

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by

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ABSTRACT

Wald's theory is used to find truncated sequential test regions for the hypergeometric distribution. These regions are then evaluated using Aroian's direct method of sequential analysis. Using this method, the important test properties (operating characteristic (OC) function, average sample number (ASN) function and the distribution of the decisive sample number (DSN)) are found exactly. The tests are compared with other similar tests (both sequential and fixed size) and estimation of the parameter after completion of the sequential test is treated. Numerical examples and general computer programs are also included.

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3	50	2	12	0.05	0.1	79
4	50	10	20	0.05	0.1	81
5	50	20	30	0.05	0.1	83
6	100	5	20	0.05	0.1	85
7	100	10	25	0.05	0.1	87
8	100	15	30	0.05	0.1	89
9	100	25	40	0.05	0.1	91
10	100	40	60	0.05	0.1	93

INTRODUCTION

This report is concerned with sequential tests of the hypergeometric distribution. The report is organized as follows. Chapter 1 reviews the relevant literature and gives some necessary background material. Chapter 2 describes the construction of the sequential test regions for the hypergeometric distribution and Chapter 3 applies the direct method of sequential analysis to the problem, allowing exact evaluation of the test properties. Both two and three decision test procedures are treated. Chapter 4 presents a variety of numerical examples, including fixed size and sequential test plans and their respective properties. In Chapter 5, the sequential test properties are compared with both fixed size tests and a binomial approximation to the hypergeometric problem. There, the superiority of the sequential procedures (with respect to average sample size requirements) is clearly shown. Chapter 6 examines a method of estimation which can be performed at the completion of a sequential hypothesis test. Computer programs developed for these procedures are given in the Appendix.

CHAPTER 1

BACKGROUND AND REVIEW OF THE LITERATURE

1.0 INTRODUCTION

This chapter introduces the hypergeometric distribution and reviews the relevant literature. The first two sections discuss this distribution and give examples of its proper use. The following section presents the development of fixed size sample tests for this distribution. These tests will be compared with the sequential tests developed here. The last section reviews some of the previous results with sequential tests of the hypergeometric distribution.

1.1 THE HYPERGEOMETRIC DISTRIBUTION

The hypergeometric distribution is the appropriate distribution when each element of a finite population can be considered either a success (defective) or a failure (non-defective) and sampling is performed without replacement from a finite population. It is assumed that there are D defectives in a population of size N. If a random sample of size n is drawn from the population, the probability that it will contain x defectives is given by the hypergeometric probability mass function:

$$h(x;N,n,D) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} = \frac{D!(N-D)!n!(N-n)!}{N!x!(D-x)!(n-x)!(N-D-n+x)!} \qquad (1.1)$$

$$\max(0,n-N+D) \leq x \leq \min(n,D)$$

The cumulative distribution function, which gives the probability that the number of defectives x found in a sample of size n is less than or equal to r is

$$P(x \leq r) = H(r;N,n,D) = \sum_{i=0}^{r} h(i,N,n,D). \qquad (1.2)$$

Lieberman and Owen (1961) give tables of h(x,N,n,D) and H(r,N,n,D) for N=1(1)50(10)100.

This dissertation considers inferences about D, the number of defectives in a finite population of known size N. Inferences concerning other parameters (e.g., the population size N) can also be made by generalizing the procedures presented here.

The hypergeometric distribution has many important applications, one of the most important being acceptance sampling. Grant (1964) gives some discussion of this along with a table of log factorials which are useful for calculating hypergeometric probabilities. The hypergeometric distribution can also be used for nonparametric tests of location and dispersion, as explained in Owen (1962) and Gibbons (1971). In addition, it is used for tests of significance for the equality of two unknown binomial proportions and for tests of independence of

two binary characteristics. These last applications arise because the hypergeometric is the null distribution for tests which can be formulated in terms of a 2x2 table in which one is testing the independence of the rows and columns.

Johnson and Kotz (1969), in their book, give some further applications of the hypergeometric distribution along with a somewhat more theoretical development of the distribution, including the moments and the generating functions. They also present discussion of different methods of fixed size sample estimation and of some extensions of the distribution. A rather complete set of references relating to the hypergeometric distribution is also included there.

1.2 WHEN THE HYPERGEOMETRIC DISTRIBUTION SHOULD PROPERLY BE USED

There is often confusion as to when the hypergeometric distribution should be used and when the binomial distribution should be used. As mentioned earlier, the hypergeometric distribution should be used when sampling without replacement from a finite population. If the population being sampled is infinite (or is large enough to be considered so) or if sampling is done with replacement, the binomial distribution should be used.

The important distinction to be made between the binomial situation and the hypergeometric situation is in the probability

of finding a defective at each inspection. An example makes this clearer. Consider a quality control engineer who wishes to make inferences about the number of defectives in a lot of 100 items. First the assumption is made that the machine which produces the items produces defects with some constant probability. If 20 items are to be inspected and they are inspected as they come from the machine (not necessarily consecutive items), the binomial distribution should be used because the probability that a given item will be defective is the same for each of the 20 items tested. On the other hand, if after the lot of 100 items has been produced, a random sample of 20 items is selected from this lot and inspected without replacement, the hypergeometric distribution is appropriate. This is because the probability of observing a defective is no longer constant for each of the items inspected.

In general, if the population is finite, tests performed without replacement using the hypergeometric distribution will be more powerful. The size of the increase in power is related to the ratio of the sample size to the population size (along with the other parameters). If this ratio is less than 0.1, the difference between the tests will in most cases be small enough to ignore. Also, if the ratio is small enough, either distribution may be used when sampling without replacement. This is of course due to the fact that the hypergeometric distribution will approach the binomial distribution if N is

increased and D/N remains constant. If, however, these above conditions are not met, one should use the hypergeometric distribution.

1.3 FIXED SIZE SAMPLE TESTS FOR THE HYPERGEOMETRIC DISTRIBUTION

By using the tables of the hypergeometric distribution, one can find the fixed size sample test which has the smallest necessary sample size and still has error probabilities which meet the desired specifications. A procedure for doing this is presented below.

In a two decision test there are two types of errors to be considered. These are shown in Figure 1.1.

Decision Based on Test Results

True State of Nature		H_0	H_a
	H_0	NO ERROR	α ERROR
	H_a	β ERROR	NO ERROR

Figure 1.1 Error Probabilities for a Two Decision Test

The first is called a Type I or α error and is made when there is a decision to accept the alternate hypothesis, H_a, when in fact the null hypothesis, H_0, is true; α usually denotes the probability of such an error. A Type II or β error occurs

when the null hypothesis is accepted when in fact the alternate hypothesis is true; β usually denotes the probability of a Type II error. The following notation, however, is used here. Let α and β denote the desired probabilities of the Type I and Type II errors respectively and let α^* and β^* denote the probabilities actually given by the fixed size sample test.

Guenther (1969) describes in detail a method of obtaining the desired test by using the tables of the hypergeometric distribution provided, for example, by Lieberman and Owen (1961). The method is straightforward and is outlined below.

The simple hypotheses to be tested are

$$\begin{aligned} H_0&: \quad D=D<N/2 \\ \text{versus} \quad H_1&: \quad D=D_1>D_0 \end{aligned} \tag{1.3}$$

The requirements of the desired test are:

$$P(\text{accept } H_0)=\begin{cases} 1-\alpha & \text{if } D=D_0 \\ \beta & \text{if } D=D_1 \end{cases} \tag{1.4}$$

This expression gives two points of the operating characteristic (OC) function. The OC function gives the probability of accepting H_0 as a function of the true state of nature. Because it is required that the OC function be nonincreasing, it is clear that $\alpha+\beta<1$. An example of a typical OC function is shown in Figure 1.2. Actually, the OC function is a step function in this case, although it is shown here as if it were continuous. This practice will be used throughout to make the graphs easier to read.

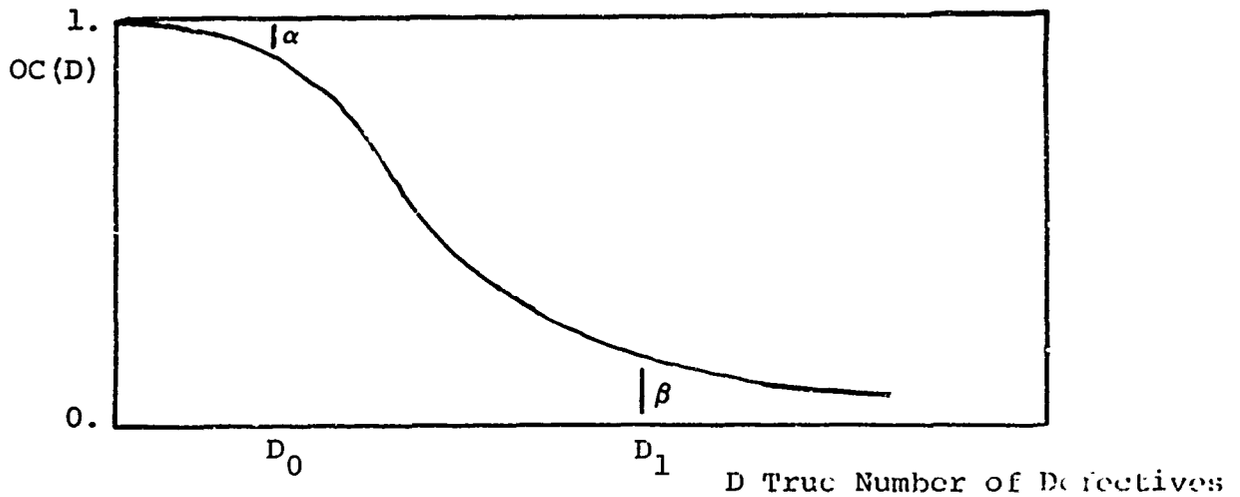

Figure 1.2 Typical OC Curve for a Two Decision Test

The fixed size sample test plan is defined by the sample size n* and a critical value c*. If the number of defectives found in the sample of size n* is greater than c*, the alternate hypothesis, H_1, is accepted; otherwise, the null hypothesis, H_0, is accepted. The procedure for finding such a test if the appropriate values of the probability function are available (from tables, for example) is given by Guenther (1969) and is repeated here for completeness.

1. Start with the critical value c*=0
2. Find the largest n, say n_L, such that $H(c^*;N,n_L,D_0) \geq 1-\alpha$. This inequality will hold for all $n \leq n_L$
3. Find the smallest n, say n_S, such that $H(c^*;N,n_S,D_1) \leq \beta$. This inequality will be satisfied for all $n \geq n_S$

4. If $n_S \leq n_L$, the plan (n_S, c^*) satisfies the requirements with minimum sample size and n_S is n^*.
5. If $n_S > n_L$, increment c^* by one and go to step 3.

After n^* and c^* have been determined, the values of the error probabilities can be determined as follows:

$$\begin{aligned} \alpha^* &= 1 - \sum_{x=0}^{c^*} h(x;N,n^*,D_0) \\ &= 1 - H(c^*;N,n^*,D_0) \\ \beta^* &= \sum_{x=0}^{c^*} h(x;N,n^*,D_1) \\ &= H(c^*;x,N,n^*,D_1) \end{aligned} \tag{1.5}$$

where $H(r;N,n,D) = \sum_{x=0}^{r} h(x;N,n,D)$, the cumulative distribution function, is given by Lieberman and Owen (1961) for N=1(1)50(10)100. For values of N not contained in the table, Guenther (1973) suggests another approach for obtaining the desired test. This approach is improved here. Using an approximation due to Wise (1954),

$$H(r;N,n,D) \approx \sum_{x=0}^{r} \binom{n}{x} \pi^x (1-\pi)^{n-x} \tag{1.6}$$

where

$$\pi = \left(D - (c/2)\right) \Big/ \left(N - \left((n-1)/2\right)\right) \tag{1.7}$$

Letting $P = 1 - H(c;N,n,D)$, it becomes necessary to find a solution for (1.6) in terms of n given N,D,c and P. Guenther (1973)

gives the approximation

$$n \approx .5\{x^2_{2c+2;1-P}(1/\pi - 0.5) + c\} \tag{1.8}$$

to be used with the binomial distribution, and solves the system of (1.7) and (1.8) iteratively, failing to realize that one can solve for the approximate n directly. It can be shown that

$$\begin{aligned} &n \approx F(N,P,c,D) \\ &\text{where} \quad F(N,P,c,D) = \{\chi^2(2N+1-D+5c) + c(D-c)\} / (4D-2c+\chi^2) \end{aligned} \tag{1.9}$$

and χ^2 is the P^{th} percentile of the chi-squared distribution with 2c+2 degrees of freedom. Further, an approximation of the desired test can be found by increasing c until the interval n

$$F(N,P,D_1,c) < n < F(N,P,D_0,c) \tag{1.10}$$

contains at least one integer. A first guess for the test is then obtained by using the smallest integer in the interval for the sample size n* and c for the critical value c*.

The computer program given in the Appendix carries out the above calculations to find an initial guess for c*, and then uses the first procedure of Guenther to find the test size n* and the critical value c* exactly.

The fixed size sample tests used for comparisons in this dissertation are not randomized. Randomization, if used, can serve two purposes. When developing tests of a discrete distribution, such as hypergeometric distribution considered here, the size of the error probabilities are often considerably less

than the desired size. Randomization can be introduced into the test such that the probability of a Type I (or Type II) error can achieve exactly the desired size, i.e., $\alpha=\alpha^*$ (or $\beta=\beta^*$). This is valuable if two alternate tests are to be compared. Also, the use of this technique may allow for a small reduction in the required sample size.

Randomization is accomplished, for example, by accepting H_0 with probability γ if c* defectives are found in the sample, H_1 being accepted with a probability $(1-\gamma)$. γ is chosen such that the probability of a Type I (or Type II) error is exactly the desired size. For simplicity of presentation, randomized tests are not used here. Most of the comparisons will not be affected by this omission.

1.4 EXTENSION TO THREE DECISION TESTS

The extension of the above results to two-sided or three decision tests* is straightforward. The hypotheses are stated as follows:

$$\begin{aligned} & H_1: \quad D=D_1<N/2 \\ \text{versus} \quad & H_0: \quad D=D_0>D_1 \\ \text{versus} \quad & H_2: \quad D=D_2>D_0 \end{aligned}$$

In this case, there are four types of errors which can be made. α_1 is the probability of accepting H_1 when H_0 is true and β_1 is the probability of accepting H_0 or H_2 when H_1 is true. α_2 is the

*The three decision test is a generalization of the standard two-sided test; that is, separate α and β errors can be specified for each alternate hypothesis (see Goss (1974b)

probability of accepting H_2 when H_0 is true and β_2 is the probability of accepting H_1 or H_0 when H_2 is true. These error probabilities are shown in Figure 1.3.

Decision Based on Test Results

True State of Nature	H_1	H_0	H_2
H_1	NO ERROR		β_1 ERROR
H_0	α_1 ERROR	NO ERROR	α_2 ERROR
H_2	β_2 ERROR		NO ERROR

Figure 1.3 Error Probabilities for a Three Decision Test

The fixed size sample test is found as in the two decision test, except that there are now four constraints to be considered:

$$\begin{aligned} &P(\text{accept } H_1) = 1-\beta_1 && \text{if } D=D_1 \\ &P(\text{accept } H_1) = \alpha_1 && \text{if } D=D_0 \\ &P(\text{accept } H_2) = 1-\beta_2 && \text{if } D=D_2 \\ &P(\text{accept } H_2) = \alpha_2 && \text{if } D=D_0 \end{aligned} \qquad (1.12)$$

This gives two points on each of two of the OC curves for the three decision test. Graphs showing typical OC functions for a three decision test are shown in Figure 1.4.

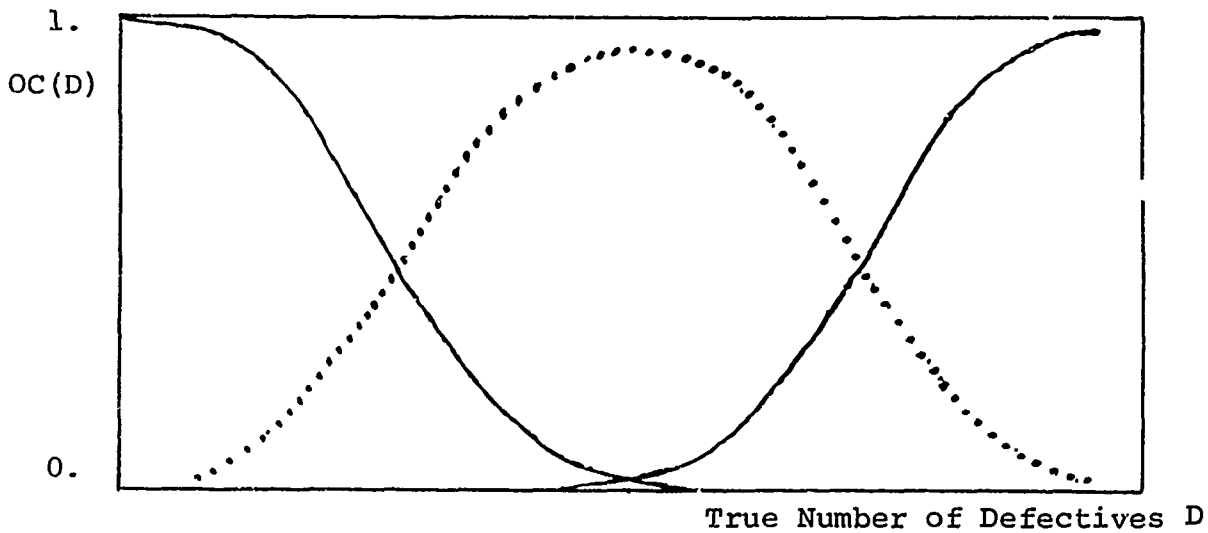

Figure 1.4 Typical OC Functions for a Three Decision Test

The solid curve, decreasing with D, gives the probability of accepting H_1 as a function of the true state of nature, D. The solid curve increasing with D gives the probability of accepting H_2. The dotted curve gives the probability of accepting H_0 for different values of D. The sum of these three curves is of course 1. Again, these curves are really step functions because of the discrete nature of the parameter D, but are shown here as being continuous.

The test plan consists of the sample size n* and two critical values c_L^* and c_U^*. The decision rules for the test are then:

$$\begin{aligned} &\text{accept } H_1 \text{ if } x \leq c_L^* \\ &\text{accept } H_0 \text{ if } c_L^* \leq x \leq c_U^* \\ &\text{accept } H_2 \text{ if } x \geq c_U^* \end{aligned} \qquad (1.13)$$

This test may also be randomized by specifying probabilities γ_1 and γ_2 for the probabilities of accepting H_0 or H_2 when D is equal to c_L^* or c_U^* respectively.

1.5 PREVIOUS RESULTS WITH SEQUENTIAL TESTS OF THE HYPERGEOMETRIC DISTRIBUTION

The first suggestion of a sequential test of the hypergeometric distribution is found in Wald (1947) where it is given as an example of a simple case of dependent observations. Here Wald shows how to perform the sequential test by using a sequential probability ratio test (SPRT). Chung (1950), Dumas (1969), Dumas (1970) and Ghosh (1970) also discuss the SPRT for this distribution. Similar tests, with some modification, are used here as explained in Chapter 2.

Chung (1950) derives approximations for the likelihood ratio used in sequential tests of the hypergeometric distribution. These approximations are valid when the proportion of defectives in the population is small. He also gives some examples of their use in acceptance sampling plans. He does not discuss the resulting test properties. Yang (1968) gives rules for a sequential test of the hypergeometric distribution, but does not mention the properties of these tests.

Dumas, in his papers, gives approximation for the boundaries of the sequential test region. These approximations are based on the use of Stirling's approximation for the factorials which appear in the equations. He also discusses the resulting shape of the regions and their geometric relation to the regions used in sequential tests of the binomial distribution. Some further comparisons between the tests of these two distributions are given in Chapter 4.

Ghosh (1970) notes that because the observations from a hypergeometric distribution are not independent, the OC function and other test characteristics will be "quite difficult to determine." This is because the approximations given by Wald (1947) hold only when the observations are independently and identically distributed (i.i.d.). Ghosh (1970) gives conjectural formulas for the OC and the average sample number (ASN) which he states might hold under certain conditions. These conjectural formulas have not been investigated. It appears that the task of solving these formulas is considerably more difficult than using Aroian's direct method of sequential analysis, as is done here. Also, the direct method uses no approximations and the results are exact

CHAPTER 2

CONSTRUCTION OF THE SEQUENTIAL TEST REGIONS FOR TWO AND THREE DECISION TEST PROCEDURES FOR THE HYPERGEOMETRIC DISTRIBUTION

2.0 INTRODUCTION

In this chapter, the method of finding a sequential test region for the hypergeometric distribution is presented. Tests are given to determine the number of defectives in a finiite population when the population size is known. In the first part of the chapter, one-sided, two decision tests are treated. In the second part, the method is extended to deal with two-sided, three decision tests. Numerical examples to illustrate both cases are provided. The third section discusses sequential tests of composite hypotheses and the resulting OC functions.

2.1 CONSTRUCTION OF THE BOUNDARIES FOR A TWO DECISION SEQUENTIAL TEST

The sequential procedures developed here will consider the situation when one is sampling items one at a time without replacement from a finite population of known size. Each item is then classified as a defect or as a non-defect. Based on the total number of defects observed, a decision is made to accept H_0, the null hypothesis, or H_1, the alternate hypothesis, or to take another observation. The procedure is easily generalized to consider group sampling (inspecting more than one item at each trial), although this is not considered here.

In order to test a one-sided hypothesis, the simple hypotheses

$$H_0: \quad D=D_0<N/2$$
$$\text{versus} \quad H_1: \quad D=D_1>D_0 \tag{2.1}$$

are specified (if $D_0>N/2$, one can reverse the designation of "defective" and "effective" observations), where D is the number of defectives in a population of size N. The sequential test procedure for distinguishing between these two hypotheses is as follows. Following Wald (1947), a sequential probability ratio test (SPRT) is carried out by calculating the likelihood ratio at each trial. Let $y_i=1$ if a defect is observed at trial i and 0 otherwise and let $f(y,D)$ denote the probability of observing a defect when there are D defects remaining in the population. The likelihood at step n with a sample outcome $x=\sum_{i=1}^{n} y_i$ is then equal to

$$L_n(x,D_i)=f(y_1,D_{i1})\cdot f(y_2,D_{i2})\ldots f(y_n,D_{in}) \tag{2.2}$$

and the likelihood ratio for the simple hypothesis in (2.1) is

$$\frac{L_n(x,D_1)}{L_n(x,D_0)} = \frac{f(y_1,D_{11})\cdot f(y_2,D_{12})\ldots f(y_n,D_{1n})}{f(y_1,D_{01})\cdot f(y_2,D_{02})\ldots f(y_n,D_{0n})} \tag{2.3}$$

Here D_{ij} is the number of defectives remaining in the population at trial j under hypothesis H_i and is dependent on what has happened in the (j-1) previous trials. Note that $D_1=D_{10}$ and

$D_0=D_{00}$. The test is then carried out using the following procedure:

$$\text{accept } H_0 \text{ if } \quad \frac{L_n(x,D_1)}{L_n(x,D_0)} \leq B,$$

$$\text{accept } H_1 \text{ if } \quad \frac{L_n(x,D_1)}{L_n(x,D_0)} \geq A, \tag{2.4}$$

$$\text{take another sample if } \quad B < \frac{L_n(x,D_1)}{L_n(x,D_0)} < A.$$

The values A and B, which are needed for the test, are quite difficult to determine exactly. However, the approximate values

$$A \simeq (1-\beta)/\alpha \qquad B \simeq \beta/(1-\alpha) \tag{2.5}$$

given by Wald (1947) serve the purpose well (this is demonstrated in the examples to follow). Here α is the desired probability of a Type I error and β is the desired probability of a Type II error, as explained in Chapter 1. It can be shown, for example, that $\alpha'+\beta' \leq \alpha+\beta$ where α' and β' are the error probabilities actually given by the SPRT.

In the case of a discrete distribution, the likelihood ratio of a given sample is independent of the particular order in which that sample is observed. Therefore, the likelihood ratio at each trial is easily computed as the ratio of the probabilities of obtaining the observed sample. In this case,

$$\frac{L_n(x,D_1)}{L_n(x,D_0)} = \frac{\binom{D_1}{x}\binom{N-D_1}{n-x}}{\binom{D_0}{x}\binom{N-D_0}{n-x}} \tag{2.6}$$

After expanding the binomial coefficients, the likelihood ratio can be expressed as

$$\frac{L_n(x,D_1)}{L_n(x,D_0)} = \frac{D_1!(N-D_1)!}{D_0!(N-D_0)!} \cdot \frac{(D_0-x)!(N-D_0-n+x)!}{(D_1-x)!(N-D_1-n+x)!} \tag{2.7}$$

where the factor on the left is independent of both the sample outcome and the trial number.

Some authors (e.g., Ghosh (1970) and Chung (1950)) have given approximations to the likelihood ratio. These approximations are seen to be unnecessary, as the expression in (2.7) is simply and efficiently evaluated by using a table look-up of log factorials within the computer program used to compute the test region. The program given in the Appendix uses this method. The log likelihood ratio is computed as

$$\ell n\left\{\frac{L_n(x,D_1)}{L_n(x,D_0)}\right\} = K(N,D_0,D_1)+\ell n\left((D_0-x)!\right)+\ell n\left((N-D_0-n+x)!\right) -\ell n\left((D_1-x)!\right) -\ell n\left((N-D_1-n+x)!\right) \tag{2.8}$$

where $K(N,D_0,D_1)=\ell n(D_1!)-\ell n(D_0!)+\ell n\left((N-D_1)!\right)-\ell n\left((N-D_0)!\right)$.
To carry out the test $\ell n\{L_n(x,D_1)/L_n(x,D_0)\}$ is computed at each trial and compared with $a=\ell n(A)$ and $b=\ell n(B)$.

There are two interesting peculiarities which arise in the treatment of sequential tests of the hypergeometric distribution. First, because we define $\binom{n}{x}=0$ if $x>n$, the likelihood ratio can take on the following values

$$\frac{L_n(x,D_1)}{L_n(x,D_0)} = \infty \qquad \text{if } x>D_0$$

$$\frac{L_n(x,D_1)}{L_n(x,D_0)} = 0 \qquad \text{if } x<n+D_1-N \tag{2.9}$$

In the first case, more than D_0 defectives have been found and therefore H_0 cannot be true and H_1 is accepted. In the second case, not enough items remain in the population to ever accept H_1; therefore, H_0 is accepted.

The other interesting characteristic of an SPRT for the hypergeometric distribution is that the test is always closed at some finite trial number. That is, there is a natural truncation point where the test is terminated. This occurs because the finite population is depleted by sampling. In cases of independent observations (e.g., the binomial or Poisson distribution), no such natural truncation exists and the test is shown to be closed only as $n\to\infty$ (Wald ,1947).

An upper bound for this natural truncation point is $N-D_1+D_0+1$, although it can be considerably less depending on the size of the desired error probabilities (α and β errors). This follows directly from (2.9). If $n=N-D_1+D_0$, there is only one value of x ($x=D_0=n+D_1-N$) such that neither of the inequalities on the right side of (2.9) is satisfied. If $n>N-D_1+D_0$ one of these inequalities will hold for each value of x and either H_0 or H_1 must be accepted.

In order to carry out the sequential procedure in practice, it is usually easier to have available upper and lower (integer)

limits on the number of defectives necessary for a decision (one way or the other) at each trial. If we let $c_L(n)$ denote the lower limit and $c_U(n)$ the upper limit, at trial n the sequential test procedure becomes:

$$\begin{aligned} &\text{accept } H_0 \quad \text{if } x \leq c_L(n) \\ &\text{accept } H_1 \quad \text{if } x \geq c_U(n) \end{aligned} \tag{2.10}$$

where x is the number of defectives observed at trial n. If neither of these inequalities holds, another sample is taken.

If possible, the values $c_L(n)$ and $c_U(n)$ are obtained by inversion of the equations

$$\begin{aligned} b &= g(x, D_0, D_i, n, N) = \ell n \left\{ \frac{L_n(x, D_1)}{L_n(x, D_0)} \right\} \\ a &= g(x, D_0, D_1, n, N) = \ell n \left\{ \frac{L_n(x, D_1)}{L_n(x, D_0)} \right\} \end{aligned} \tag{2.11}$$

by solving for x. The values c_L and c_U are then expressed as

$$\begin{aligned} c_L(n) &= \left[g^{-1}(b, D_0, D_1, n, N) \right] \\ c_U(n) &= \left[g^{-1}(a, D_0, D_1, n, N) \right] + 1 \end{aligned} \tag{2.12}$$

where $K = [R]$ signifies the largest integer value K such that $K \leq R$ and g^{-1} is the inverse function of g when solving for x. Because of the factorials in the function g, these functions must be inverted numerically. This is a simple procedure because at each step one has very close lower bounds for the

new critical values. That is, the value of the inverse function

$$x=g^{-1}(b,D_0,D_1,n-1)\leq g^{-1}(b,D_0,D_1,n). \qquad (2.13)$$

Also, the values in (2.13) are generally very close together. As a result, each critical value in (2.12) is usually obtained with only one or two evaluations of the log likelihood ratio.

The critical values $c_L(n)$, $c_U(n)$, $n=1,2,\ldots n^{**}$ define the critical regions for the test, where n^{**} denotes the natural truncation point. An example of such a region is shown graphically in Figure 2.1. Note that at trial n, if $c_L(n)<0$, no decision in favor of H_0 is possible and if $c_U(n)>n$, no decision for H_1 is possible.

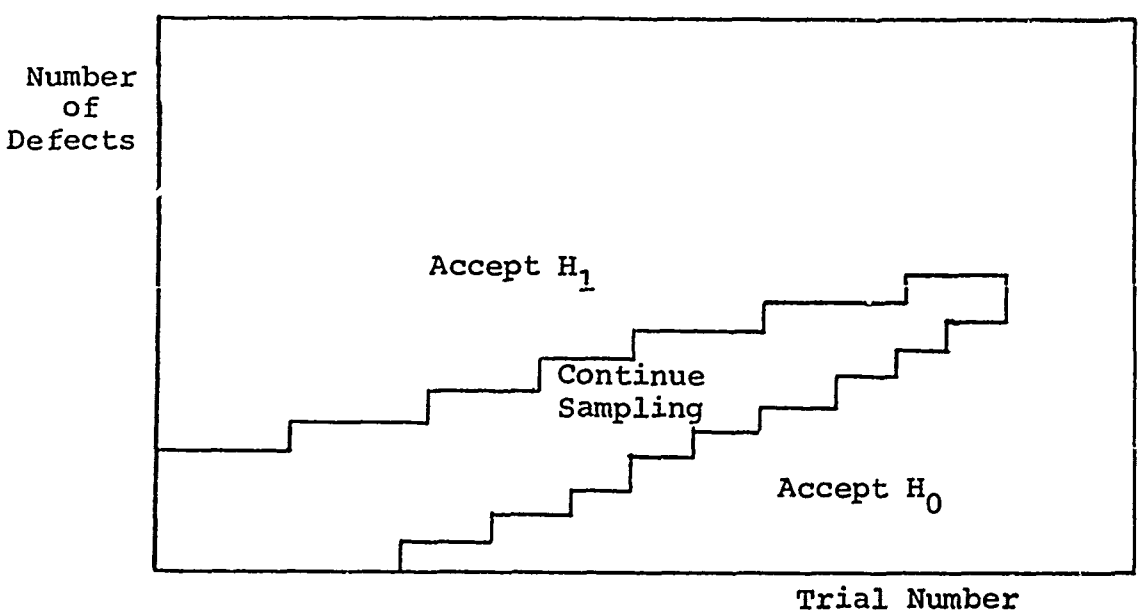

Figure 2.1 Typical Two Decision Test Region for the Hypergeometric Distribution

A computer program given in the Appendix carries out the above calculations to compute the desired sequential test regions for any desired values of N, D_1, D_0, α and β.

Often n**, the natural truncation point, is much larger than n*, the sample size of the corresponding fixed size sample test. This undesirable situation indicates that with positive probability, the test will require samples which exceed the necessary sample size of the corresponding fixed size sample test. Because this probability is rather small and because this probability (of such a large sample) is largest when the parameter being tested is actually in the "indifference range" ($D_0<D<D_1$), the test can usually be improved by truncation at some point n_0 n*. More complete treatment of this subject, along with suggested truncation procedures, is given in Chapter 3.

A numerical example of the above procedure for determining the sequential test region is now given. The hypotheses to be tested are

$$\begin{aligned} &H_0: \quad D=D_0=25 \\ \text{versus } &H_1: \quad D=D_1=40 \end{aligned} \tag{2.14}$$

with a population size N=100. The desired error probabilities are α=0.05 and β=0.10. The following quantities are computed.

$$\begin{aligned} b=\ell n(B)&=\ell n\left(0.1/(1-0.05)\right) &&=-2.25129 \\ a=\ell n(A)&=\ell n\left((1-0.1)/0.05\right) &&=2.89037 \\ K(N,D_0,D_1)&=\ell n(40!)-\ell n(25!)+\ell n\left((100-40)!\right)-\ell n\left((100-25)!\right) \\ &= &&-10.9452 \end{aligned} \tag{2.15}$$

As an example, we compute the critical limits at trial n=32. The log likelihood ratio at trial 32 with x defectives observed in the sample is

$$\ell n\left\{\frac{L_{32}(x,D_1)}{L_{32}(x,D_0)}\right\} = K(100,25,40)+\ell n\Big((25-x)!\Big)+\ell n\Big((75-32+x)!\Big) - \ell n\Big((40-x)!\Big)-\ell n\Big((60-32+x)!\Big) \qquad (2.16)$$

and is tabulated for different values of x in Table 2.1.

Table 2.1
Log Likelihood Ratio
for Different Values of x
at Trial 32

x	$\frac{L_{32}(x,D_1)}{L_{32}(x,D_0)}$
7	-3.26
8	-2.31
9	-1.34
10	-0.34
11	0.68
12	1.72
13	2.80
14	3.91
15	5.08

It is easily seen from this table that for values of $x \geq 14$, the log likelihood ratio is greater than $a=\ell n(A)$ and therefore such points belong in the accept H_1 region. For values of $x \leq 8$, the log likelihood ratio is less than $b=\ell n(B)$ and therefore these points belong in the accept H_0 region.

When the above procedure is carried out for each trial, one obtains the critical values needed for the sequential test.

This was done for the example and the results are given in Table 2.2. A graphical presentation of the region is given in Figure 2.2. One should notice that at trial 67, $c_L(n)+1=c_U(n)$. This implies that a decision must be made for either H_0 or H_1 and that the test will not continue past this point. This is the natural truncation point mentioned earlier. Note that this value is considerably smaller than n**, the upper limit $(N-D_1+D_0+1=86)$ on the natural truncation point.

2.2 CONSTRUCTION OF THE BOUNDARIES FOR A THREE DECISION SEQUENTIAL TEST

In this section, the procedure for developing three decision sequential tests for the number of defectives in a finite population of size N is given. Three decision tests are often necessary in practice. This is also true for acceptance sampling. One example of their use would be when one must distinguish among lots of items which are of superior quality (for which some incentive bonus might be given), standard quality and substandard quality. The tests given below are suitable for such applications.

A discussion of the previous work concerning sequential three decision tests is deferred until the end of this section, after which the reader will be more familiar with the subject.

The method used for determining the test procedure is a direct extension of the method given for the two decision test presented in the first part of this chapter. The numerical example given previously will be extended to the three decision case.

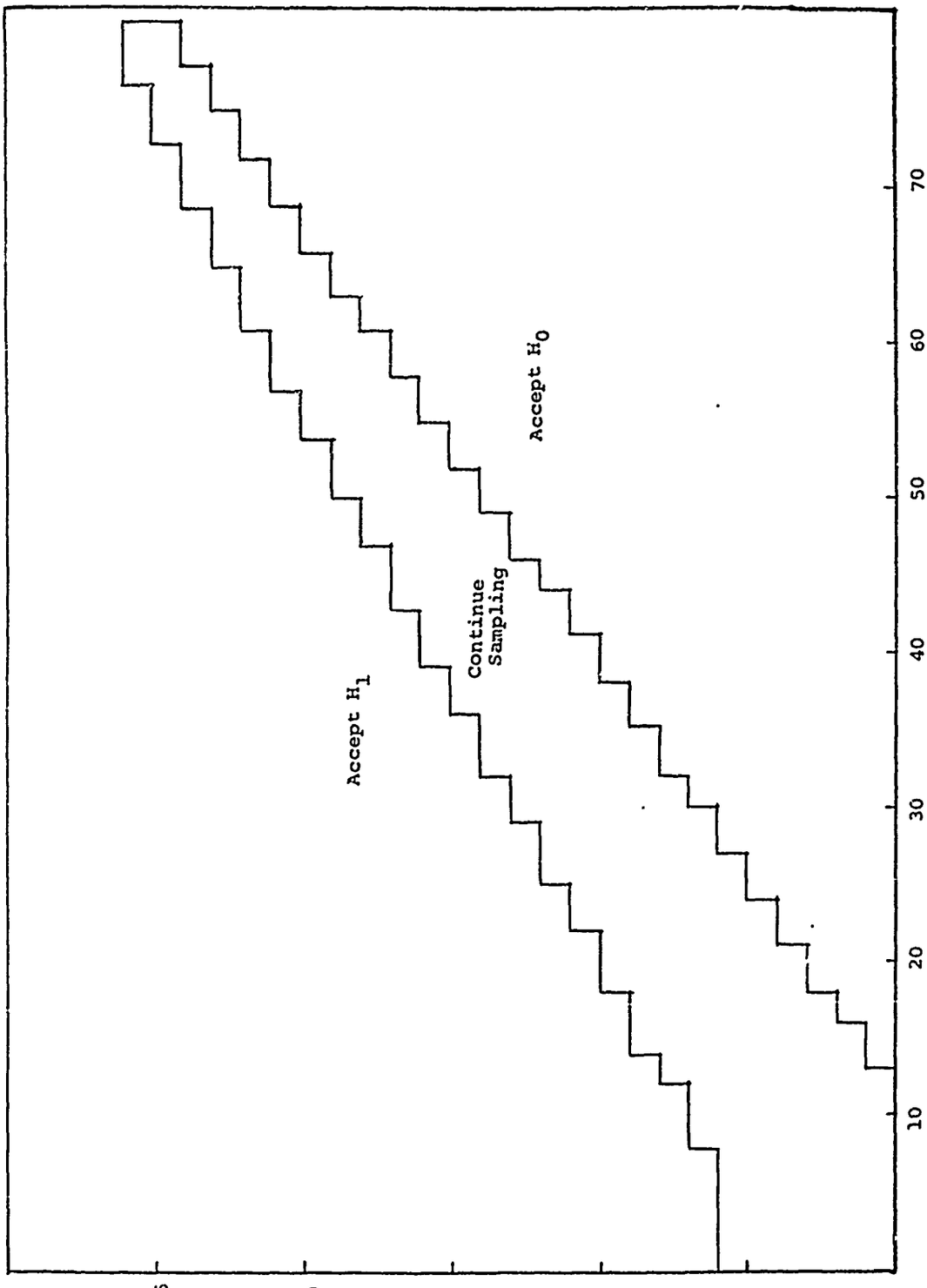

Figure 2.2 Sequential Test Region for the Two Decision Example

Table 2.2
Critical Values for the
Two Decision Example

Trial n	$c_L(n)$	$c_U(n)$	Trial n	$c_L(n)$	$c_U(n)$
1	*	*	42	11	16
2	*	*	43	11	17
3	*	*	44	12	17
4	*	*	45	12	17
5	*	*	46	13	17
6	*	6	47	13	18
7	*	7	48	13	18
8	*	7	49	14	18
9	*	7	50	14	19
10	0	7	51	14	19
11	0	8	52	15	19
12	0	8	53	15	19
13	1	8	54	15	20
14	1	9	55	16	20
15	1	9	56	16	20
16	2	9	57	16	21
17	2	9	58	17	21
18	3	10	59	17	21
19	3	10	60	17	21
20	3	10	61	18	22
21	4	10	62	18	22
22	4	11	63	19	22
23	4	11	64	19	22
24	5	11	65	19	23
25	5	12	66	20	23
26	5	12	67	20	23
27	6	12	68	20	23
28	6	12	69	21	24
29	6	13	70	21	24
30	7	13	71	21	24
31	7	13	72	22	24
32	8	14	73	22	25
33	8	14	74	22	25
34	8	14	75	23	25
35	9	14	76	23	25
36	9	15	77	23	26
37	9	15	78	24	26
38	10	15	79	24	26
39	10	16	80	24	26
40	10	16	81	25	26
41	11	16			

For a three decision test the hypotheses are specified as

$$\begin{aligned} H_1: &\quad D=D_1<N/2 \\ \text{versus } H_0: &\quad D=D_0>D_1 \\ \text{versus } H_2: &\quad D=D_2>D_0 \end{aligned} \tag{2.17}$$

where D is the number of defectives in a population of size N. At each trial, an item is selected at random without replacement from the population and classified as either a defect or a non-defect and one of four actions is taken. Either one of the three hypotheses is accepted or another sample is taken. It is the purpose of this section to develop the rules for carrying out such a test.

To devise a three decision test, a modified version of the Sobel-Wald procedure (Sobel and Wald, 1949) is used. Following their treatment, two SPRTs are used simultaneously. One SPRT, say SPRT1, is used to distinguish between H_0 and H_1. The other SPRT, say SPRT2, is used to distinguish between H_0 and H_2. As explained in Chapter 1, there are now four types of errors to be concerned with. As before, these errors are denoted α_1, β_1 (for SPRT1) and α_2, β_2 (for SPRT2). These two SPRTs are used to derive the three decision test as follows. At each step, calculate the two likelihood ratios and follow the rules:

accept H_1 if

$$\frac{L_n(x,D_0)}{L_n(x,D_1)} \leq B_1 \quad \text{and} \quad \frac{L_n(x,D_2)}{L_n(x,D_0)} \leq B_2$$

accept H_2 if

$$\frac{L_n(x,D_0)}{L_n(x,D_1)} \geq A_1 \quad \text{and} \quad \frac{L_n(x,D_2)}{L_n(x,D_0)} \geq A_2 \tag{2.18}$$

accept H_0 if

$$\frac{L_n(x,D_0)}{L_n(x,D_1)} \geq A_1 \quad \text{and} \quad \frac{L_n(x,D_2)}{L_n(x,D_0)} \leq B_2$$

otherwise, another sample is taken. Here again the values

$$\begin{aligned} A_1 &\simeq (1-\alpha_1)/\beta_1 & A_2 &\simeq (1-\beta_2)/\alpha_2 \\ B_1 &\simeq \alpha_1/(1-\beta_1) & B_2 &\simeq \beta_2/(1-\alpha_2) \end{aligned} \tag{2.19}$$

are used to approximate the true values A_1, B_1, A_2, and B_2 necessary for the test. These approximations, as will be seen from the numerical results in the next chapter, are satisfactory and provide good tests.

When carrying out the sequential test in practice, it is usually easier to have available .itical limits on the number of defectives necessary to accept one of the hypotheses at each trial. For a two-sided test, we must specify four critical limits, two for each SPRT at each trial. Let $c_L(n)$ and $c_U(n)$ denote the lower and upper limits respectively for SPRT1 at trial n. Also, let $d_L(n)$ and $d_U(n)$ denote the same critical limits for SPRT2.

The test procedure then becomes

$$\begin{aligned} &\text{accept } H_1 \text{ if} && x \le c_L(n) \text{ and } x \le d_L(n), \\ &\text{accept } H_0 \text{ if} && c_U(n) \ge x \ge d_L(n) \\ &\text{accept } H_2 \text{ if} && x \ge c_U(n) \text{ and } x \ge d_U(n) \end{aligned} \tag{2.20}$$

and otherwise take another sample.

Following the same procedure given in the first part of this chapter, these critical values are found by inverting the likelihood ratio equations and may be expressed as

$$\begin{aligned} c_L(n) &= \left[g^{-1}(b_1, D_1, D_0, n, N)\right] \\ c_U(n) &= \left[g^{-1}(a_1, D_1, D_0, n, N)\right] + 1 \\ d_L(n) &= \left[g^{-1}(b_2, D_0, D_2, n, N)\right] \\ d_U(n) &= \left[g^{-1}(a_2, D_0, D_2, n, N)\right] + 1 \end{aligned} \tag{2.21}$$

where $a_1 = \ell n(A_1)$, $b_1 = \ell n(B_1)$, etc. and the other notation is the same as that used in (2.12). The critical limits for a typical three decision test are shown graphically in Figure 2.3.

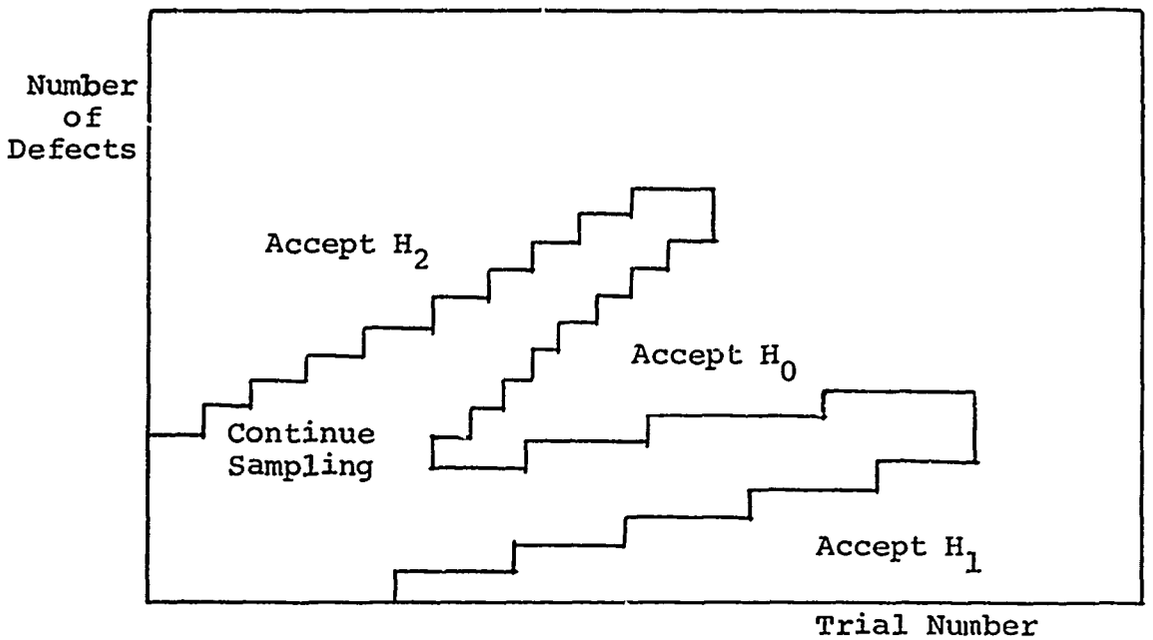

Figure 2.3 Typical Three Decision Test Region for the Hypergeometric Distribution

A computer program which is given in the Appendix calculates these critical points separately for each SPRT. The region thus found can be evaluated using the method and computer program described in Chapter 3. The special characterisitics of the two decision test, such as the natural truncation point, are also present in the three decision test. Truncation of these tests is considered in the next chapter.

We now extend the example of the test given in (2.14) and consider a three decision test. There, the hypotheses considered were

$$H_0: \ D=D_0=25$$
$$\text{versus } H_1: \ D=D_1=40 \qquad (2.22)$$

Now the hypotheses being tested are:

$$\begin{aligned} H_1&: \ D=D_1=10 \\ \text{versus } H_0&: \ D=D_0=25 \\ \text{versus } H_2&: \ D=D_2=40 \end{aligned} \tag{2.23}$$

again with a population of size N=100. The desired error probabilities are

$$\alpha_1=0.05,\ \beta_1=0.10,\ \alpha_2=0.05 \text{ and } \beta_2=0.10$$

We first obtain the values

$$\begin{aligned} b_1&=\ell n(B_1)=\ell n(0.05/(1-0.1)) = -2.89037 \\ a_1&=\ell n(A_1)=\ell n((1-0.05)/0.1) = 2.25129 \\ b_2&=\ell n(B_2)=\ell n(0.1/(1-0.05)) = -2.25129 \\ a_2&=\ell n(A_2)=\ell n((1-0.1)/0.05) = 2.39037 \end{aligned} \tag{2.24}$$

We now calculate the limits in (2.21) for trial n=52. The values of the log likelihood ratio for the two SPRTs are given in Table 2.3. It can be seen from this table that one should accept H_1 if $x\leq 7$, accept H_0 if $10\leq x\leq 15$ and accept H_2 if $x\geq 19$, otherwise another item is inspected. If this procedure is carried out for each trial up to n**, the critical values for the three decision test given in Table 2.4 will be obtained. A graphical presentation of the test regions for this test is given in Figure 2.4.

Table 2.3
Log Likelihood Ratio
for Different Values of x
at Trial 52

x	$\frac{L_{52}(x,D_0)}{L_{52}(x,D_1)}$	$\frac{L_{52}(x,D_2)}{L_{52}(x,D_0)}$
6	-5.46	-14.12
7	-3.50	-12.85
8	-1.13	-11.58
9	1.21	-10.31
10	4.36	- 9.05
11	∞	- 7.77
12	∞	- 6.48
13	∞	- 5.18
14	∞	- 3.85
15	∞	- 2.48
16	∞	- 1.08
17	∞	0.37
18	∞	1.88
19	∞	3.47
20	∞	5.15

The following is a brief sketch of the different approaches to three decision tests which have been treated in the literature. The discussion here is general in that it pertains to no specific distribution. Ghosh (1970) and Goss (1974b) give excellent and somewhat more comprehensive treatment of this subject. No attempt has been made to cover the many applications of these tests. For this, the reader is referred to Wetherill (1966).

Wald (1947), in his book, gives a method of formulating a two-sided test by using weight functions. Barnard (1947), in his review of Wald's book, mentions an alternate method which simply tests the null hypotheses separately, against the two alternatives. This is done by using two SPRTs at one time. The resulting test regions are shown geometrically in Figure 2.5.

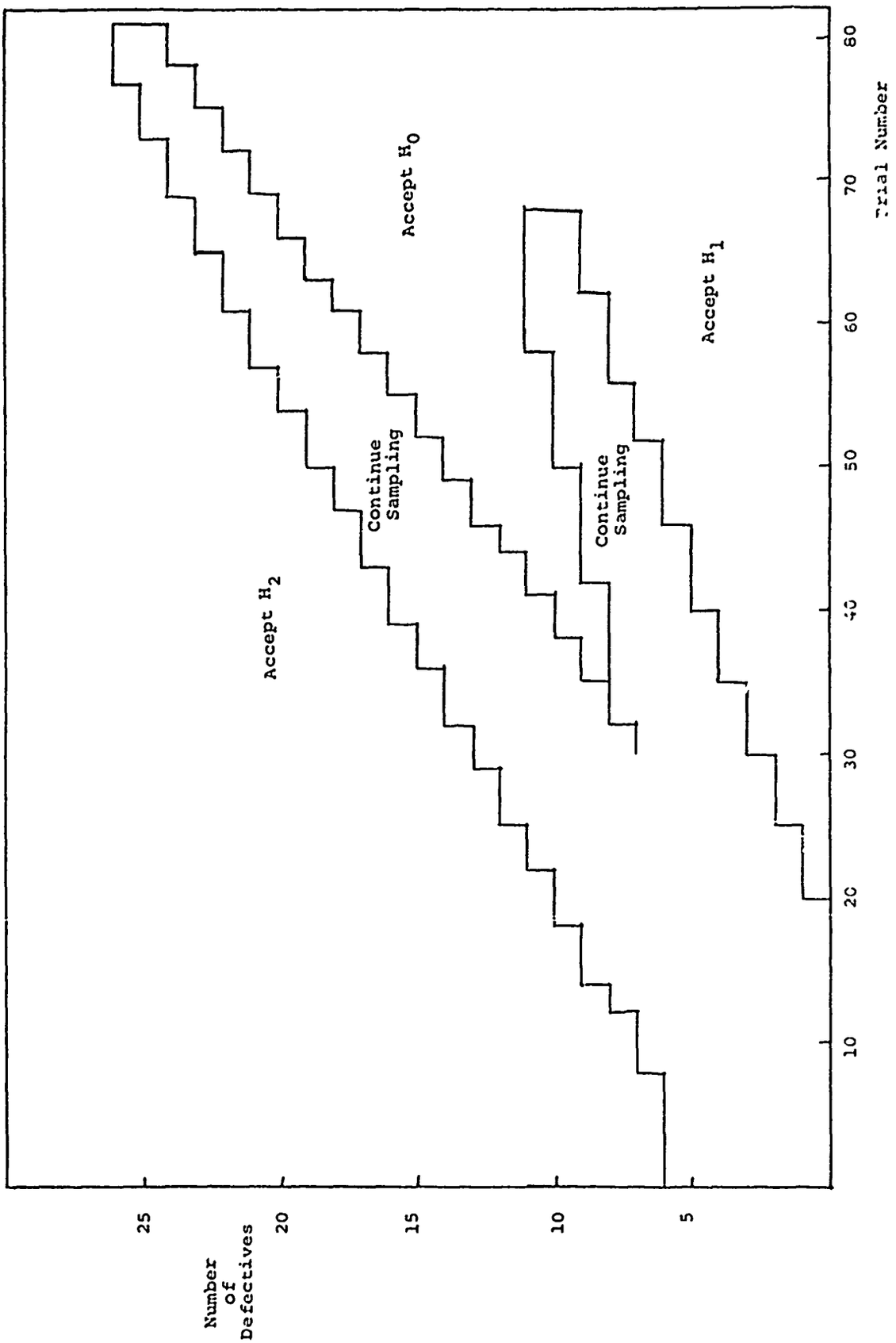

Figure 2.4 Sequential Test Region for the Three Decision Example

Table 2.4
Critical Values for the
Three Decision Example

Trial	n	$c_L(n)$	$c_U(n)$	$d_L(n)$	$d_U(n)$	Trial	n	$c_L(n)$	$c_U(n)$	$d_L(n)$	$d_U(n)$
1		*	*	*	*	42	5	9	11	16	
2		*	*	*	*	43	5	9	11	17	
3		*	3	*	*	44	5	9	12	17	
4		*	3	*	*	45	5	9	12	17	
5		*	3	*	*	46	6	9	13	17	
6		*	3	*	6	47	6	9	13	18	
7		*	4	*	7	48	6	9	13	18	
8		*	4	*	7	49	6	9	14	18	
9		*	4	*	7	50	6	10	14	19	
10		*	4	0	7	51	7	10	14	19	
11		*	4	0	8	52	7	10	15	19	
12		*	4	0	8	53	7	10	15	19	
13		*	4	1	8	54	7	10	15	20	
14		*	5	1	9	55	7	10	16	20	
15		0	5	1	9	56	8	10	16	20	
16		0	5	2	9	57	8	10	16	21	
17		0	5	2	9	58	8	11	17	21	
18		0	5	3	10	59	8	11	17	21	
19		0	5	3	10	60	8	11	17	21	
20		1	5	3	10	61	8	11	18	22	
21		1	6	4	10	62	9	11	18	22	
22		1	6	4	11	63	9	11	19	22	
23		1	6	4	11	64	9	11	19	22	
24		1	6	5	11	65	9	11	19	23	
25		2	6	5	12	66	9	11	20	23	
26		2	6	5	12	67	9	11	20	23	
27		2	6	6	12	68	10	11	20	23	
28		2	7	6	12	69			21	24	
29		2	7	6	13	70			21	24	
30		3	7	7	13	71			21	24	
31		3	7	7	13	72			22	24	
32		3	7	8	14	73			22	25	
33		3	7	8	14	74			22	25	
34		3	7	8	14	75			23	25	
35		4	8	9	14	76			23	25	
36		4	8	9	15	77			23	26	
37		4	8	9	15	78			24	26	
38		4	8	10	15	79			24	26	
39		4	8	10	16	80			24	26	
40		5	8	10	16	81			25	26	
41		5	8	11	16						

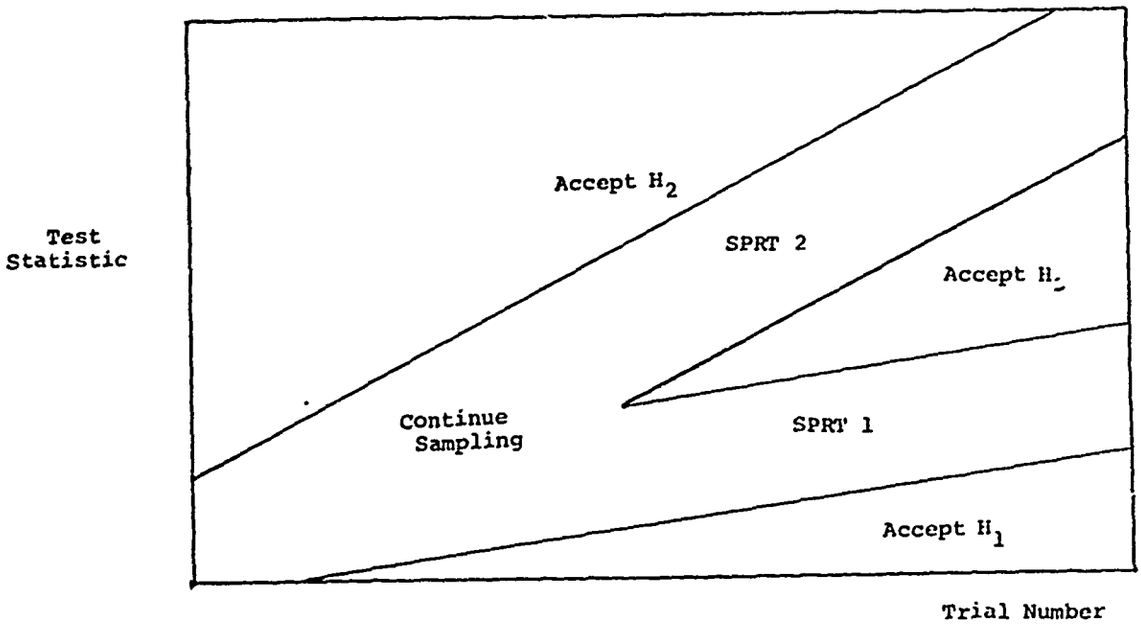

Figure 2.5 A Three Decision Sequential Test Region

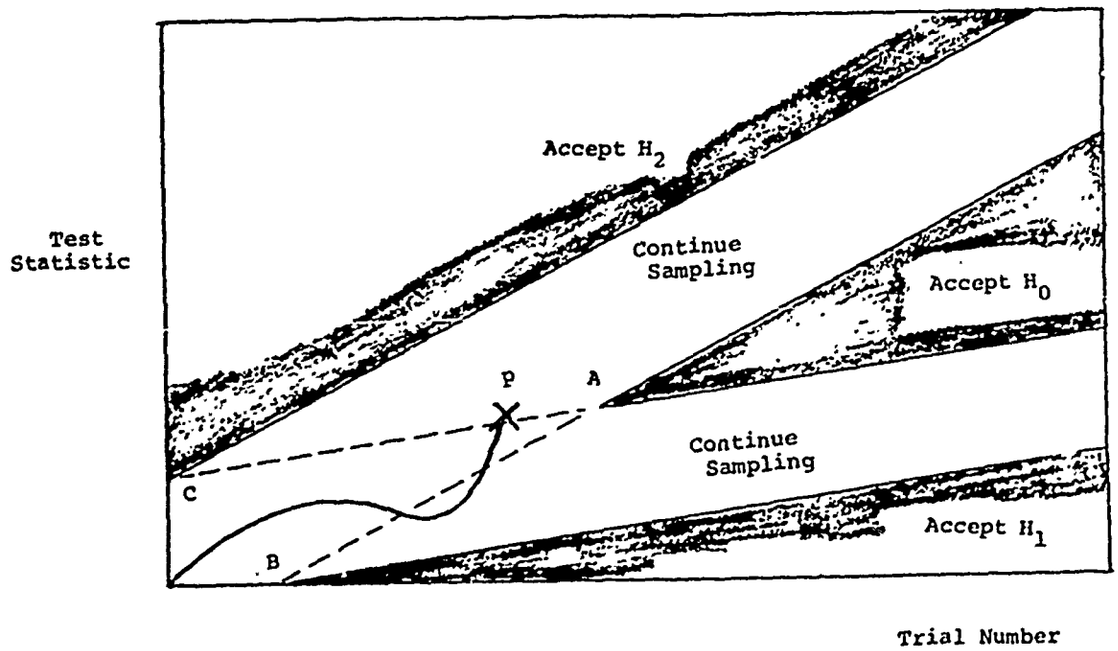

Figure 2.6 Illustration of the Independence of the Sobel-Wald Sequential Test Region

Sobel and Wald (1949), in their paper, treat the three decision test in detail. They use a test similar to that suggested by Barnard. The difference is that each SPRT is treated independently of the other. This would mean, for example, that when line AB is crossed by the path shown in Figure 2.6, we no longer allow acceptance of H_1 and concern ourselves only with the results of SPRT2. Thus, H_0 is accepted when line AC is crossed at point p, before a shaded region is even reached. Sobel and Wald hasten to point out that such a test, which depends not only on the total sample results, but also on the sample path (order of the observations), cannot be an optimal one. However, the test was used in their case because the independence of the two tests enabled the authors to derive approximations for some of the properties of this three decision test. The Sobel-Wald tests and their approximate properties are treated in detail by Ghosh (1970). Here, we use the direct method of sequential analysis which can be used to find the exact properties of any specified sequential test region.

Goss (1974b), when treating three decision sequential tests of the mean of a normal distribution, compared the Sobel-Wald test with the Barnard test. He used the direct method to obtain exact test results for such tests. From his results, (as one would expect intuitively) it is seen that the test with independently run SPRTs has a smaller expected sample size, but slightly larger error probabilities. The differences, however, are quite small. For this reason and because it has somewhat more intuitive appeal,

the approach suggested by Sobel and Wald is used here, with the modification that a decision to accept a hypothesis is allowed if and only if one of the conditions in (2.20) is satisfied; that is, if and only if one enters a shaded region in Figure 2.6.

Another approach to the three decision test is given by Armitage (1950). In this paper, Armitage suggests using three SPRTs simultaneously. The three SPRTs are constructed to distinguish between H_1 and H_0, H_2 and H_0 and between H_1 and H_2. This is shown graphically in Figure 2.7.

In tests where $\alpha_1+\alpha_2<\beta_1+\beta_2$, (which is the case in most practical applications), the test will be almost identical to the Sobel and Wald type regions used above. If, however, $\alpha_1+\alpha_2>\beta_1+\beta_2$, regions such as the ones shown in Figure 2.8 are obtained. In such cases, the method of Armitage might be worth using. The test regions would be similar to those shown in Figure 2.7. The computer program given in the Appendix is general and may be used to evaluate such regions if desired.

2.3 TEST OF COMPOSITE HYPOTHESES AND THE OC FUNCTION

This section will consider sequential tests of composite hypotheses. It will be shown here that the Wald SPRT, used in Sections 2.1 and 2.2 and based on pairs of simple hypotheses, can be used to obtain satisfactory sequential tests for composite hypotheses. The discussion below pertains to two decision tests, although the ideas also apply to $k>2$ decision tests.

When finding a fixed size sample test to choose between one of two specified hypotheses, one must specify both the sample size

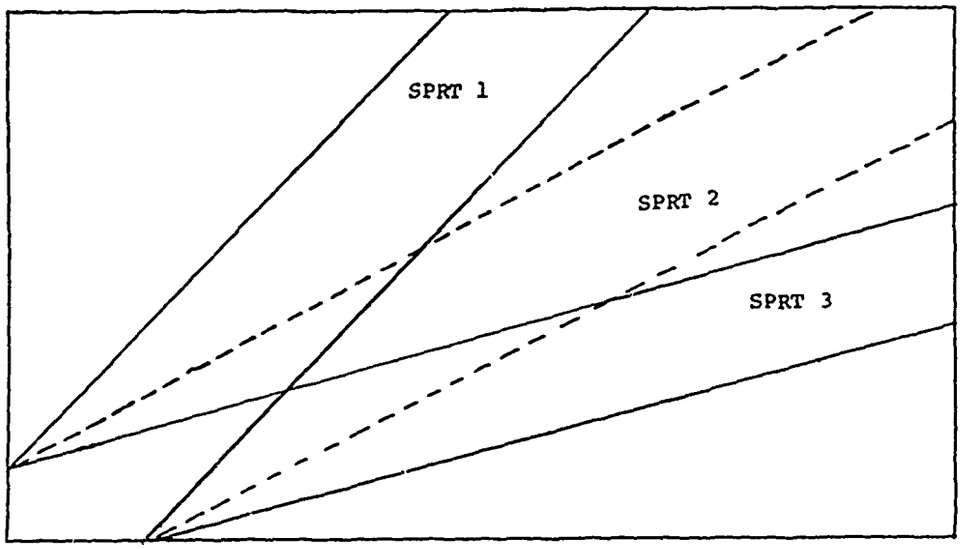

Figure 2.7 Armitage's Sequential Test Region

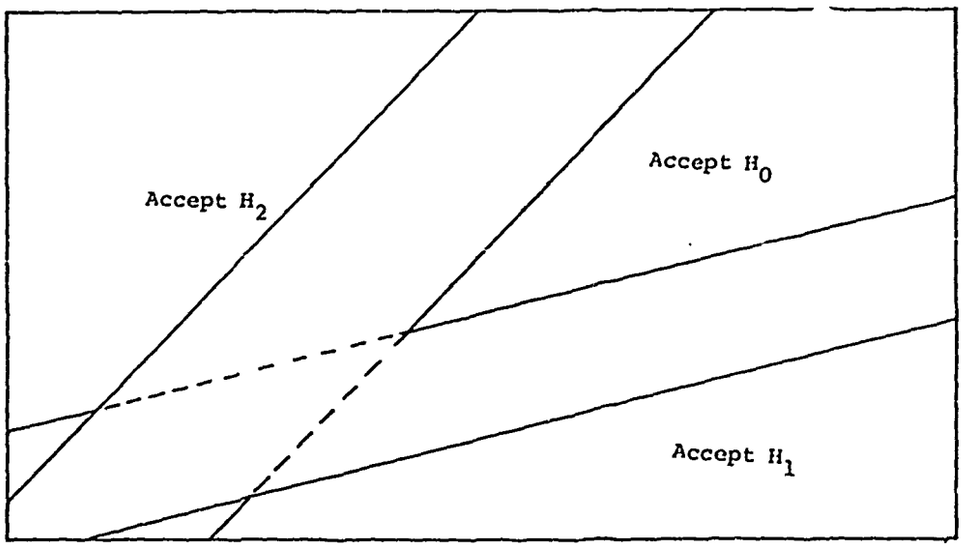

Figure 2.8 Sobel-Wald Region when $\alpha_1 + \alpha_2 > \beta_1 + \beta_2$

n* and critical value c* to give the desired error probabilities, as explained in Section 1.3 (randomization, of course, can also be used in the test). When this special case is generalized to a sequential procedure where stopping rules are selected for each trial, the problem of selection of the proper test becomes much more complicated because there are many more possible tests to choose from. To find a sequential test, one must choose an upper and a lower limit for the number of defectives at each trial number and possibly n_0, a truncation point for the sequential test.

It is well known that the Wald SPRT gives optimum regions for testing a simple hypothesis against a simple alternative under certain conditions (Wald and Wolfowitz, 1948). Such hypotheses are stated, for example, as

$$\begin{aligned} H_0&: \quad D=D_0 \\ \text{versus } H_1&: \quad D=D_1 \end{aligned} \tag{2.25}$$

as shown in Figure 2.9. The hypotheses are represented as points if they are simple, as in this case, and as line segments if they

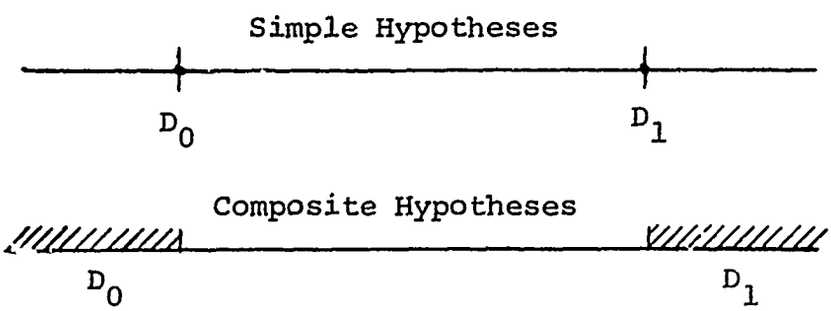

Figure 2.9 Simple and Composite Hypotheses

are composite. For our purpose, we define simple and composite hypotheses to be hypotheses with exactly one point, and more than one point respectively. Statistical tests between two alternative simple hypotheses imply that the experimenter believes that there are only two possible values for the true state of nature. Such situations do not often occur in practice.

In most cases the hypotheses to be tested are composite and expressed in a form similar to

$$H_0: D=D_0 \quad \text{versus} \quad H_1: D\neq D_0 \tag{2.26}$$

$$\text{or} \quad H_0: D\leq D_0 \quad \text{versus} \quad H_1: D\geq D_1>D_0 \tag{2.27}$$

When using a statistical test, the important distinction between the simple hypotheses in (2.25) and the composite hypotheses of (2.26) and (2.27) is that in the latter one is interested in all of the points of the OC function over a specified range of the parameter values given by the hypothesized states of nature.

The hypotheses shown in (2.26) do not contain any specific alternative and are the types generally specified in so-called "tests of significance." Users of such tests generally use a specified significance level (α error) and sample size, but do not mention a specific alternative hypothesis and therefore often do not consider the "power" of their tests. The rationale for such a test is that there is a strong prior belief in (or preference for) the null hypothesis and that it is not to be rejected unless there is strong evidence (i.e., at the $1-\alpha$ confidence level) that it is not true.

By examining the Type II error (which is one minus the power of the test at a specified alternative) one can determine if the significance level of the test has been set too low (or too high) for a given sample size or if the sample size is much too large (or too small) for the required sensitivity against alternatives to the null hypothesis. Either of these consequences could be costly. It does no harm for even the "significance tester" to investigate to which alternatives his test will be sensitive. From this it is seen that it is important to examine the power of a statistical test.

In this light, the pair of hypotheses in (2.27) is considered. Here a range of values has been specified for H_1, the alternative hypothesis, as well as for H_0, the null hypothesis (see Figure 2.9). The values in between D_0 and D_1 constitute an "indifference zone." For the situation where one must make a decision for either H_0 or for H_1, and where there are positive costs (tangible or not) for both types of errors, this is a more practicable way of specifying the hypotheses to be tested.

This again brings out the subtle difference between a "test of significance" and other composite tests of hypotheses. A test of significance might be valid, for example, for a test used in proving some law of nature, for which it is nearly impossible to specify all of the possible alternatives. In contrast, when testing the ability of a new drug to cure a disease, for example, the situation is different.

If the proportion of successful cures of a drug is to be compared with that of a control or a placebo, the hypotheses to

be tested will usually be stated as $H_0: p_1=p_2$ vs. $H_1: p_1<p_2$, where p_1 and p_2 are the probabilities of a successful cure for the control and the drug being tested respectively (both beirg unknown). In this case, there are true costs (although they are probably intangible) for both types of errors; that is, for accepting the new drug as "significantly better"* when it is not and for rejecting it when it is "significantly better." Because both of these errors are important, it is imperative that the experimenter examine the power of his statistical test so that the errors can be balanced if necessary. These same ideas are important in the development of sequential tests of composite hypotheses.

When developing sequential tests, it is usually necessary to specify some specific alternative(s) to the null hypothesis. This is so that proper stopping rules can be formulated to control both types of errors and so that the test properties of the sequential test can be assessed. If one wishes to test a composite hypothesis such as (2.27), we must find a sequential test procedure which has a satisfactory OC function over a specified range of parameter values. This is usually done with respect to some additional criterion concerning the cost of sampling.

Although the Wald procedure provides optimal tests under certain conditions, there remains the problem of finding optimum** sequential tests for the composite hypotheses considered here.

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*Here mean a difference of practical significance, rather than simp a difference of statistical significance.

**The criterion for optimality is left open for now. More treatment is given to this subject in Section 3.2.

In Sections 2.1 and 2.2 sequential test regions were found by specifying simple hypotheses. Wald (1947) discusses this problem at some length. He comes to the conclusion that the test of the simple hypotnesis in (2.25) can be used to approximate a test of a composite hypothesis such as (2.27) without much loss of efficiency. This is the method most commonly used to find regions for a sequential test of a composite hypothesis.

One should examine the possible consequences of using such an approximation, that is, carefully examine the OC function of the test. If the resulting OC function is not close to the desired OC function, the test region can be modified so that it is. The numerical examples given in Chapters 4 and 5 will show how this is done by, for example, comparing the OC function of a fixed size test with that of a sequential test. Although no claim of optimality is made for the above tests, a procedure for finding such optimal or near optimal tests is outlined in Section 3.2.

CHAPTER 3

EVALUATION OF THE TEST REGIONS USING THE DIRECT METHOD OF SEQUENTIAL ANALYSIS

3.0 INTRODUCTION

This chapter describes the evaluation of the sequential test regions for the hypergeometric distribution. In the first section, the direct method of sequential analysis is introduced. It is this method which is used to find the exact properties of the sequential test regions given in Chapter 2. Section 3.2 explains how truncation of the regions can be used to improve the properties of a sequential test and suggests procedures for doing this. The following two sections explain in detail how the direct method is used to obtain the test properties for the two and three decision sequential test regions developed in Chapter 2. Numerical examples for each of these cases are also given.

3.1 THE DIRECT METHOD OF SEQUENTIAL ANALYSIS

The direct method of sequential analysis, given by Aroian (1968), describes a general method whereby the exact properties of a given sequential test region may be obtained. Since Aroian's 1968 article, the method has been used in a variety of applications, including tests for the mean of a normal distribution with the standard deviation known (Aroian and Robison, 1969), and unknown (Schmee, 1974); two-sided tests of the normal distribution with the standard deviation known (Goss, 1974b),

sequential rank tests (Elfring and Schultz, 1973); tests of the binomial distribution (Corneliussen and Ladd, 1970 and 1971) and tests of the variance of a normal distribution with mean known or unknown (Aroian, Gorge, Goss and Robison, 1975).

Before using a sequential test procedure, one should know or have available reasonable approximations to the actual test properties. The most important test properties are the true α and β error probabilities (denoted α' and β' here) and the expected or average sample number (ASN), which is a function of the true state of nature. A typical ASN function is shown in Figure 3.1. Also of interest is the operating characteristic (OC) function which gives the probability of accepting H_0 as a function of the true state of nature (in this case, D, the actual number of defectives in the population). A typical OC function is shown in Figure 3.2. The true α and β error probabilities for a two decision test are obtained directly from the OC function as

$$\begin{aligned} \alpha' &= 1-OC(D_0) \\ \beta' &= OC(D_1) \end{aligned} \qquad (3.1)$$

Approximations to the OC and ASN functions are given by Wald (1947). These approximations are valid only if the observations are independent. This is not the case with the hypergeometric distribution. Also, even if the restriction does hold, the adequacy of these approximations varies from test to test. The direct method of sequential analysis as explained below, will allow one to find both the OC and ASN functions exactly.

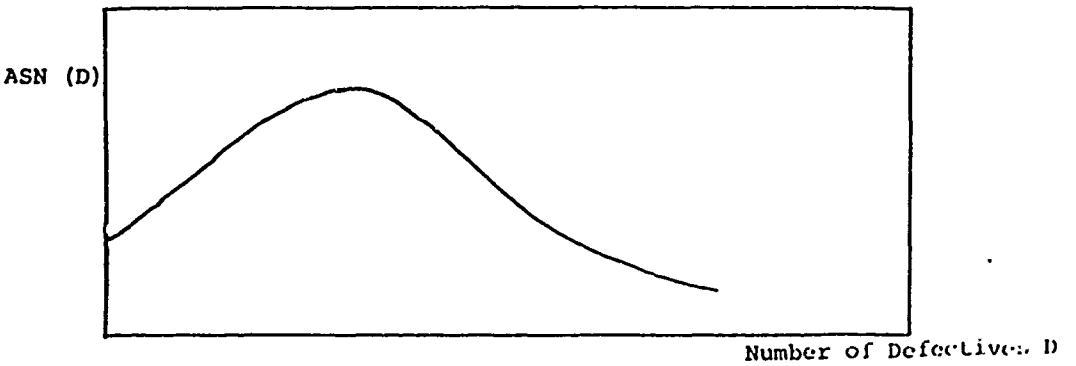

Figure 3.1 Typical ASN Function

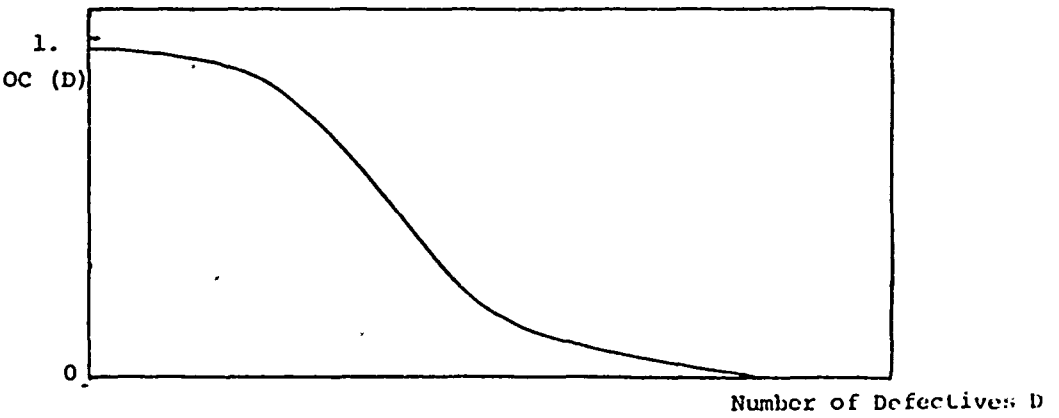

Figure 3.2 Typical OC Function

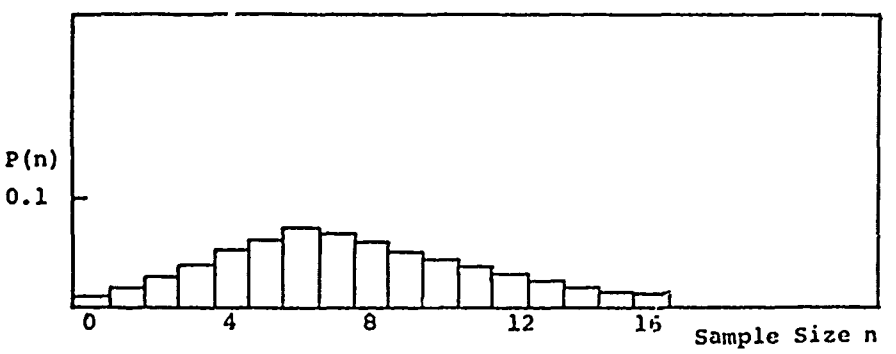

Figure 3.3 Typical Distribution of the DSN

Another interesting test characteristic, which is often neglected, is the distribution of the decisive sample number (DSN), that is, the probability mass function of the sample size required to come to a decision. This distribution is also a function of the true state of nature. From this distribution, one can obtain the ASN, the variance of the sample number (VSN) or other moments. The direct method is also used to find the distribution of the DSN. A typical probability mass function for the DSN is shown in Figure 3.3.

In general, the direct method is carried out as follows. Once the sequential test region has been specified, one then chooses a state of nature which allows the computation of the probability of accepting each possible hypothesis at the first trial. The remaining probability, that is, the probability of being in the continuation region, is spread out among all of the possible values of the sample statistic which are included in the continuation region. At the second trial, another sample is taken. It is again necessary to find the probability of accepting each hypothesis and the distribution of probability of remaining in the continuation region. Using convolutions, one can continue this process at each succeeding trial until the test is truncated or until the probability of continuation is so small as to be insignificant. This procedure is then repeated using different values for the state of nature, each giving a point on the OC function and a distribution of the DSN. This is done for the hypergeometric distribution in Sections 3.3 and 3.4.

3.2 TRUNCATION OF THE SEQUENTIAI TEST REGION

One disadvantage of using sequential test procedures is that because the sample size is a random variable, it is sometimes possible (usually with small probability) for the sample size to be significantly larger than the sample size necessary for a fixed size sample test (n*). This section presents methods for truncating sequential tests at some trial, say n_0. This will result in a closed sequential test whose test properties, with respect to the ASN function, will be much improved. The price paid for this improvement, as shown in Chapter 4, is usually quite small.

Wald and Wolfowitz (1948) show that a SPRT for a simple hypothesis with a simple alternative and with i.i.d. observations, has the smallest ASN (at the parameter values specified by the simple hypotheses) of all other tests with the same α and β error probabilities. This of course also implies that the ASN of the SPRT will be smaller than the sample size of the corresponding fixed size sample test. It must be remembered, however, that this is guaranteed to be true only for the parameter values specified by the simple hypotheses. For parameter values which fall between these two values, the value of the ASN of the untruncated test may even rise above n*. This means that for some values of the true state of nature, the ASN of the sequential test will be greater than the sample size required for the fixed size test. It should be noted that this occurs for those values of

the parameter which are in the so-called "indifference range." This is shown in a graph of a typical ASN function in Figure 3.1. Truncation of the sequential test at some trial number, say n_0, is often used to help alleviate these problems. It has been shown and will be further demonstrated here that truncation of sequential tests will both eliminate the possibility of an extremely large sample and significantly reduce the ASN over the space of the parameter value.

It is to be expected that some price must be paid for this improvement in the test, which is indeed the case here. After truncation of a Wald-type SPRT, the true α and β error probabilities will usually increase somewhat. (In fact, the entire OC function will change.) This increase is usually quite small because the probability of such large samples is highest near the middle of the "indifference range" and relatively small near the values of the parameters specified in the simple hypotheses. Because the true α and β error probabilities of the untruncated SPRT are often smaller than the specified error probabilities (i.e., $\alpha'<\alpha$ and $\beta'<\beta$), a small increase in these probabilities can usually be tolerated. Further modification of the region near the truncation trial number (n_0) can be used to adjust these probabilities to be quite close to their desired values.

Often when truncation procedures are put forward, the truncation point suggested is from 1.5 to 3 times n* (e.g., Wald, (1947)). This is probably because in the past, very little was known about the exact properties of such untruncated tests. When

using the direct method, however, this presents no problem because the direct method is general and can be used to evaluate any specified test region. From the numerical examples given in Chapter 4, it will be seen that the truncation point can be moved much closer to n*, while still keeping $\alpha' \approx \alpha$ and $\beta' \approx \beta$. Procedures for truncating sequential tests of the hypergeometric distribution and for comparing alternate tests are discussed next.

Truncated sequential tests presented here will be truncated at n* (i.e. $n_0=n^*$). If the desired error probabilities cannot be achieved when truncating at n*, and if it is necessary to do so, n_0 can be moved one way or the other to help achieve the desired error probabilities (e.g., n_0 should be increased to decrease the true error probabilities).

Once the trial number where the test is to be truncated has been specified (i.e., n_0), it becomes necessary to determine what shape the test region should have around n_0. This has been a much debated topic and is treated at some length by Goss for the three decision normal distribution sequential test (Goss, 1974b) and for the binomial distribution sequential test (Goss, 1974a). There he compares and contrasts the "right angle" truncation and the "wedge" type of truncation illustrated in Figure 3.4. He finds that while the differences are small, the wedge truncation has a slightly lower ASN and slightly larger error probabilities.

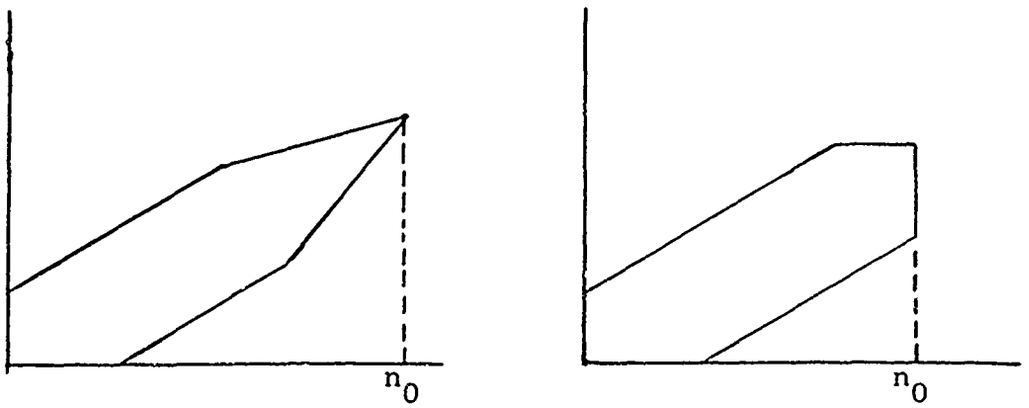

Figure 3.4 Wedge and Right Angle Truncation

When comparing different types of truncation, or to be more general, different types of test regions, one usually specifies some sort of optimality criteria with which alternative tests (i.e., different test regions) can be compared. This can also be used to optimize a test procedure. Two examples of such criteria are:

1) Minimizing the maximum of the ASN function over the parameter space.
2) Minimizing the expected sample number using a prior distribution for the parameter(s) being estimated.

Monahan (1973) uses Bayesian decision theory with a specified loss function to construct "admissible" truncated sequential test regions for testing the mean of a normal distribution.

When comparing alternate tests, however, it is usually necessary to take into account the differences in the error probabilities (or the OC function). This can be done in one of two ways. First, the randomization scheme described in Section 1.3 can be applied to the sequential test procedure and/or modification of the regions can be used to adjust the error probabilities. Also, an objective function can be constructed which takes the size of the error probabilities (or to be more general, the OC function) into consideration. The comparisons made in Chapter 5 will be made between tests which have approximately the desired α and β probabilities. In most cases this will not adversely affect the validity of the comparisons which are made.

When truncating a discrete distribution such as the binomial or the hypergeometric considered here, there are only a finite number (which may be rather large) of sensible regions to use for a given test procedure. In the truncated regions considered here, the Wald regions developed in Section 2.3 are used with the truncation rules given next.

With sequential tests of the hypergeometric (or any other) distribution, one should truncate the test such that there are no points in the continuation region from which only one decision can be made. Such points can only increase the ASN function and do not affect the error probabilities and therefore should be made part of the region for accepting the appropriate hypothesis. Point p in Figure 3.5a is an example of such a point. If the above rule is followed when truncating a typical Wald region,

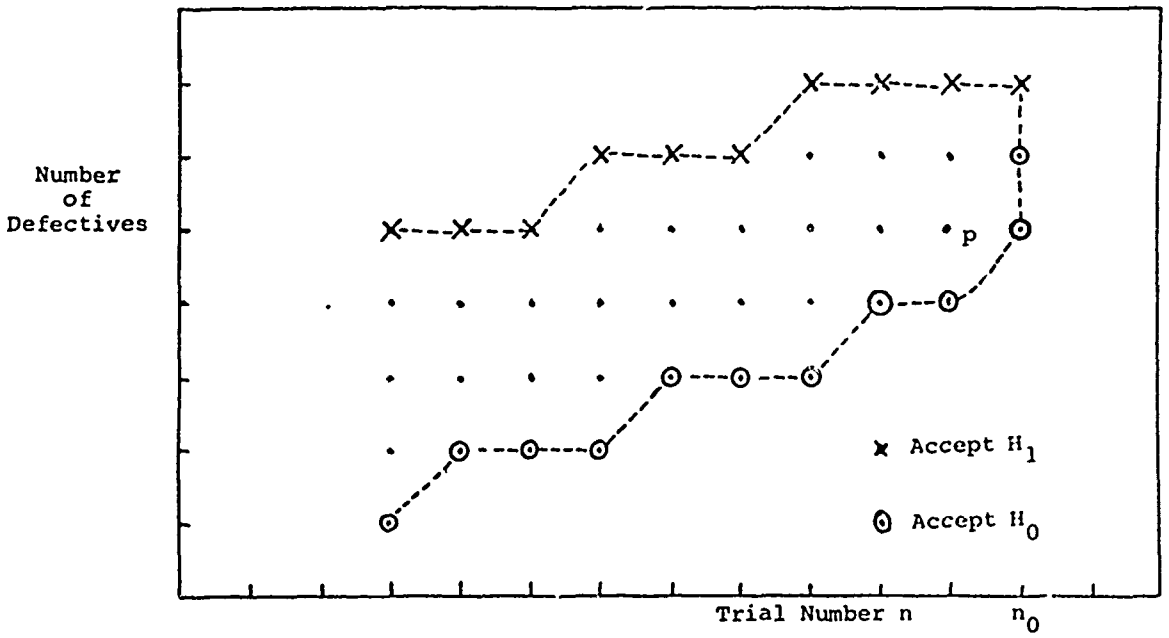

Figure 3.5a Improper Truncation

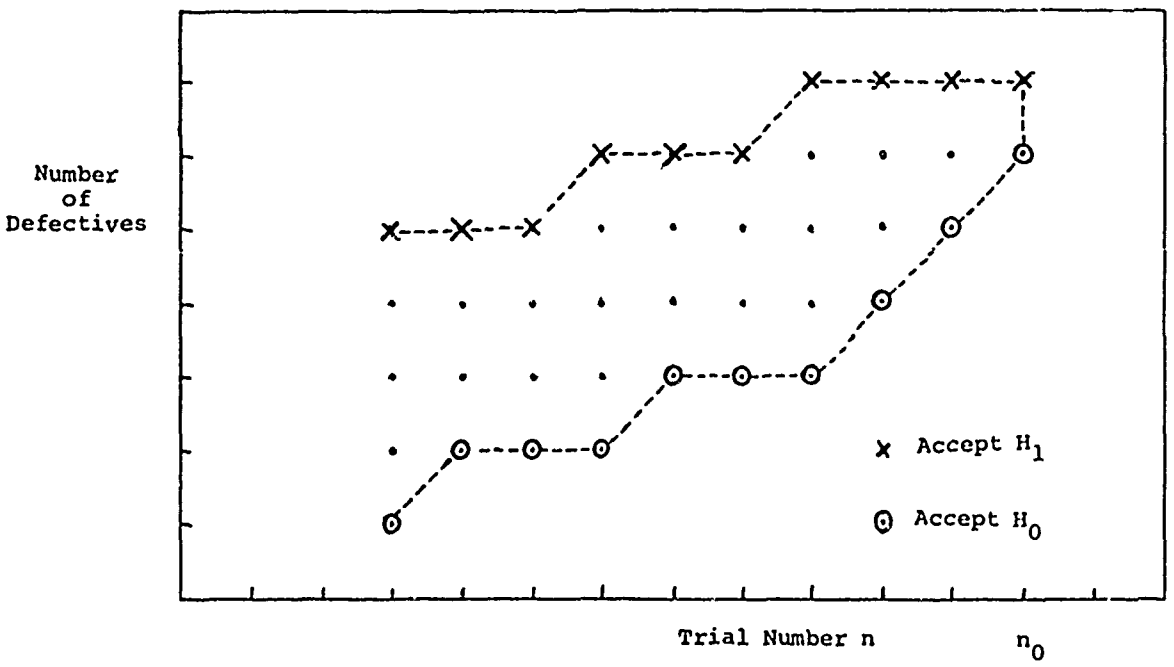

Figure 3.5b Proper Truncation

the continuation region will have a horizontal upper boundary and 45° lower boundary. This is shown in Figure 3.5b. One should note that this is the only sensible method of truncation for dichotomous distributions. When truncating a sequential test for a continuous distribution, the solution to this problem is not so clear, as there are an infinite number of points in the sample space. The problem of "optimal truncation" needs to be more fully explored for these cases.

When truncating a sequential test of the hypergeometric distribution, there are two decisions which must be made. First, one must decide the trial number where the test is to be truncated. As mentioned earlier, this can usually be at or near n^*, the sample size of the fixed size sample test. This truncation point is denoted n_0. In addition, it is also necessary to specify the critical value for the sequential test at this trial. That is, one must determine the proper $c_U(n_0)=c_L(n_0)+1$. Using the chosen n_0 and $c_U(n_0)$ and the rules given above with the Wald regions given in Section 2.1, the truncated test procedure is completely specified.

It may, under certain circumstances, be desirable to further modify the Wald regions. This can be done if one follows the rules:

$$\begin{aligned} c_U(n+1) &= c_U(n) \\ &\quad \text{or } c_U(n)+1 \\ c_L(n+1) &= c_L(n) \\ &\quad \text{or } c_L(n)+1 \end{aligned} \tag{3.2}$$

which are a generalization of the truncation rules given above and should be true of all critical values which define the test region. These rules simply state that the upper and lower boundary points do not decrease with n and never increase by more than one at any given trial.

In order to make the changes suggested above, one should know the effect of the different types of modification. These are outlined in Table 3.1.

Table 3.1
Effects of Region
Modification*

		OC(D)	ASN(D)
increase	$c_U(n)$	decrease	increase
decrease	$c_U(n)$	increase	decrease
increase	$c_L(n)$	increase	decrease
decrease	$c_L(n)$	decrease	increase

*Note, for example, if an increase is indicated in this table, the function in some cases will remain the same, but will not decrease.

The regions obtained by using the above procedure must ultimately be judged on the basis of their exact test properties, which can be found by using the direct method as shown in the next section. This usually leads to an iterative procedure to find the proper test. Such a procedure begins by evaluating a suggested test region to find its test properties. If the test properties are not satisfactory, the test region is modified using the suggestions above and evaluated again using the direct

method. Several such iterations may be necessary to achieve the desired test properties. This procedure is illustrated with the numerical examples presented in Chapter 4.

When truncating a k>2 decision test, it is necessary to truncate each of the SPRTs separately. This is illustrated in the example given in Section 3.4.

3.3 OBTAINING THE TEST PROPERTIES OF A TWO DECISION TEST REGION

This section will explain how the direct method of sequential analysis is used to find the exact test properties for a two decision test of the hypergeometric distribution. It will be shown how one can obtain both the OC function and the distribution of the decisive sample number (DSN). From these, one can also find the average sample number (ASN) and the true α and β error probabilities, α' and β'. The two decision test region developed in Section 2.1 will be evaluated here as a numerical example.

As explained in Section 3.1, the direct method is used by computing both the probability of making each decision and the distribution of probability remaining in the continuation region at each trial. The probabilities at trial n+1 are computed by convoluting the probability remaining in the continuation region at trial n with the sample taken at trial n+1. This is done for each trial for $n=1, 2, \ldots, n_0$. In a discrete distribution such as the hypergeometric, this entails summing probabilities at each trial of the test. For the hypergeometric distribution, this is

illustrated with the grid shown in Figure 3.6. The critical values which define this test region are given in Table 3.2.

For illustrative purposes, this region has been truncated at trial 10, using the rules given in Section 3.2. That is, the critical values $c_L(n)$ and $c_U(n)$ are non-decreasing with n and never increase by more than one at any given trial. The probability of reaching each point which is in the region or on its boundary is a function of the true state of nature (true number of defectives in the population, D). These probabilities differ from those of the corresponding hypergeometric distribution (i.e., the probability with a fixed size sample test) only because of the difference in the number of paths available to reach a given point. The probabilities are computed recursively starting with the point at which the origin, when no samples have to be observed; the probability of this point is, of course, 1. The recursive formula used to compute the probabilities at each trial is:

$$P(x,D,n+1,N)=I(x,n)P(x;D,n,N)\cdot(N-n-D+x)/(N-n)$$
$$+I(x-1,n)P(x-1,D,n,N)\cdot(D-x+1)/(N-n) \qquad (3.3)$$

where

$$P(x,D,0,N) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$I(x,n) = \begin{cases} 1 & c_L(n) \le x < c_U(n) \\ 0 & \text{otherwise} \end{cases}$$

The indicator function accounts for the fact that the test terminates when one of the critical points is reached.

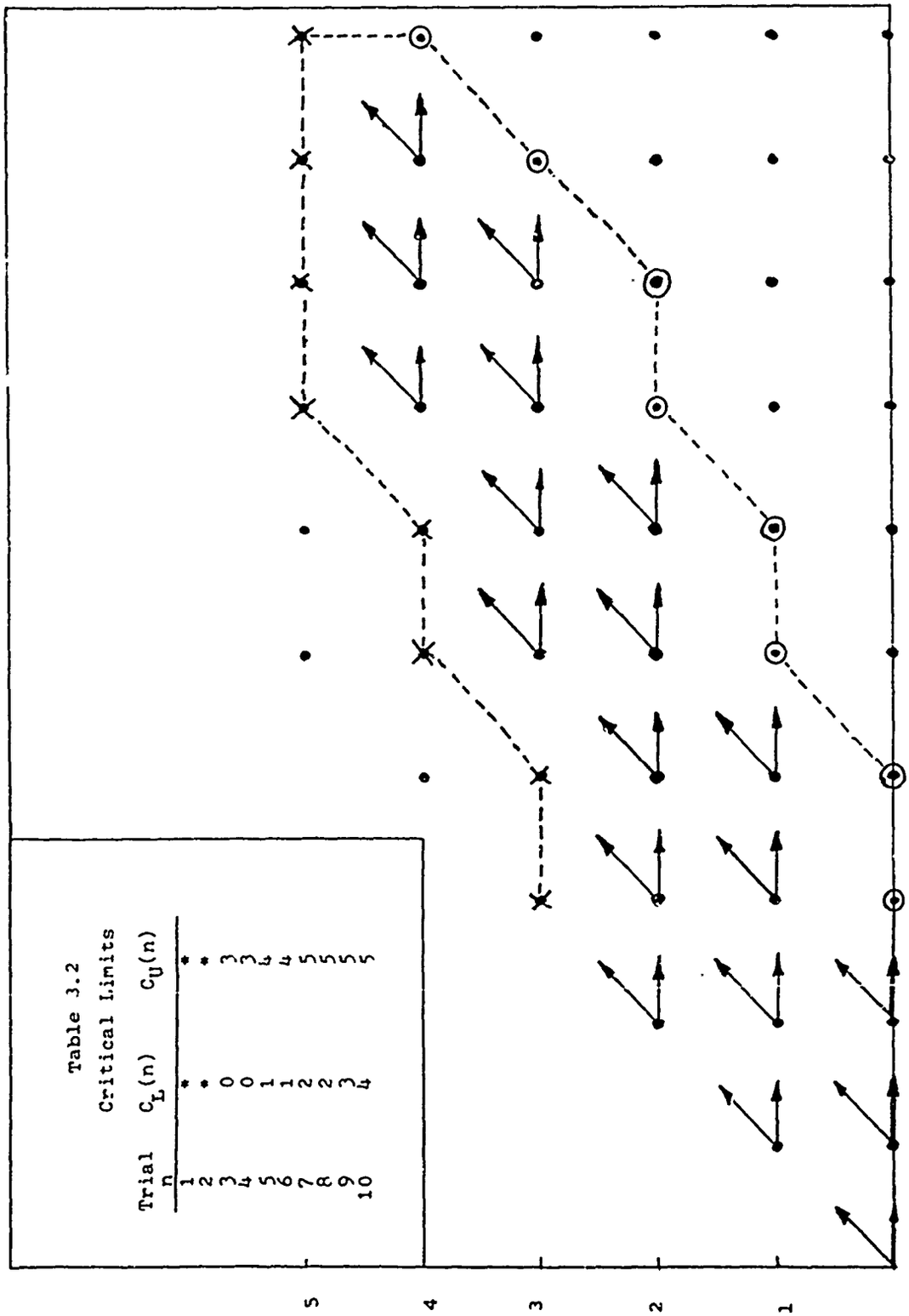

Table 3.2
Critical Limits

Trial n	$C_L(n)$	$C_U(n)$
1	*	*
2	*	*
3	0	3
4	0	3
5	1	4
6	2	5
7	2	5
8	3	5
9	4	5
10		

Figure 3.6 Probability Grid for a Two Decision Sequential Test

There are two possible outcomes at the next trial from each point in the continuation region; each inspected item may be either a defect or a non-defect. This is also illustrated in Figure 3.6. When a critical point $c_L(n)$ or $c_U(n)$ is reached at trial n, the test stops and a decision is made in favor of either H_0 or H_1. Thus, the probability that the test terminates at trial n is the probability of reaching one of the critical points at trial n.

Referring to Figure 3.6, an ✕ at a point indicates a critical point for acceptance of H_1 ($c_U(n)$); an ⊙ signifies a critical point for acceptance of H_0 ($c_L(n)$). One should note that some critical values can never be reached (e.g., 0 defectives observed at trial 4 and 4 defectives observed at trial 5). This is because there are no paths leaving the critical points.

After the probabilities for the points shown on the grid in Figure 3.6 have all been computed, it is an easy matter to determine the exact test properties. Let $A0_n$, $A1_n$ and C_n denote the events of accepting H_0, accepting H_1 and continuing to trial n+1 respectively. The probabilities of $A1_n$ and $A0_n$ at each trial are then the probabilities of reaching points $c_U(n)$ and $c_L(n)$ respectively.

This can also be expressed as follows:

$$P(Ai_n,D) = \sum_x J_i(x,n)P(x;D,n,N) \qquad (3.4)$$

where $$J_i(x,n) = \begin{cases} 1 & \text{if } (x,n) \in Ai_n \\ 0 & \text{otherwise} \end{cases}$$

Of course, $P(C_n,D) = P(C_{n-1},D) - P(A0_n,D) - P(A1_n,D)$.

Another interesting way to look at the direct method is to consider the sequential process a Markov chain with absorbing states. Each point in the sample space is a state and each boundary point is an absorbing state. The transition probabilities from each state are a function of the true state of nature. In order to use the direct method, it becomes necessary to find the probability of absorption in each of the absorbing states. This will give $P(A0_n;D)$ and $P(A1_n;D)$ for $n=0,1,\ldots n_0$. Madsen (1974) uses a Markov chain technique to find the properties of a truncated SPRT.

The distribution of the decisive sample number (DSN) (i.e., the probability of the test terminating at trial n) can be expressed as follows:

$$\begin{aligned} P(n;D) &= P(A0_n \cup A1_n;D) \\ &= P(A0_n;D)+P(A1_n;D) \end{aligned} \tag{3.6}$$

This is computed for each n up to n_0, the first trial where $c_L(n)+1=c_U(n)$. This is the truncation point of the test. The ASN and VSN are then computed as

$$ASN(D) = \sum_{n=1}^{n_0} nP(n;D) \tag{3.7}$$

$$VSN(D) = \sum_{n=1}^{n_0} (n-ASN(D))^2 P(n;D). \tag{3.8}$$

The ASN can also be expressed as

$$ASN(D) = 1 + \sum_{n=1}^{n_0-1} P(C_n;D) \tag{3.9}$$

This alternate form is given by Aroian (1975) and shows how the ASN function "builds up" at each trial of the sequential test.

The computer program in the Appendix computes these quantities. If desired, the k^{th} moment about the origin can be computed as

$$E(n^k,D) = \sum_{n=1}^{n_0} n^k P(n;D) \tag{3.10}$$

The OC function of the test is computed as

$$OC(D) = \sum_{n=1}^{n_0} P(A0_n;D) \tag{3.11}$$

As mentioned earlier, the exact α and β errors are

$$\alpha' = 1 - OC(D_0)$$
$$\beta' = OC(D_1). \tag{3.12}$$

The OC function is also calculated by the computer program given in the Appendix.

The above properties have been computed for the test region obtained in Section 2.1 and truncated as in Section 2 of this chapter. Because of space limitations, the distribution of the DSN is shown for only one value of D. These are shown in Tables 3.3 and 3.4. Graphs of the OC and ASN functions for the test are shown in Figures 3.7 and 3.8. These properties are typical of most sequential tests of the hypergeometric distribution. A complete discussion of these test properties is deferred until Chapters 4 and 5, when a more complete examination of some numerical examples is presented.

Table 3.3
Distribution of the DSN for the Two Decision Example

NUMBER OF DEFECTIVES = 30

TRIAL	P(H0)	P(H1)	P(T)	P(C)	TRIAL	P(H0)	P(H1)	P(T)	P(C)
6	0.00000	0.00050	0.00050	0.99950	44	0.03031	0.00365	0.03396	0.30477
7	0.00000	0.00000	0.00000	0.99950	45	0.00000	0.00749	0.00749	0.29728
8	0.00000	0.00057	0.00057	0.99893	46	0.04055	0.01136	0.05191	0.24537
9	0.00000	0.00194	0.00194	0.99699	47	0.00000	0.00000	0.00000	0.24537
10	0.02292	0.00442	0.02737	0.96962	48	0.00000	0.00370	0.00370	0.24167
11	0.00000	0.00000	0.00000	0.96962	49	0.03089	0.00746	0.03836	0.20331
12	0.00000	0.00216	0.00216	0.96746	50	0.00000	0.00000	0.00000	0.20331
13	0.03453	0.00519	0.03971	0.92775	51	0.00000	0.00262	0.00262	0.20065
14	0.00000	0.00000	0.00000	0.92775	52	0.02629	0.00559	0.03188	0.16877
15	0.00000	0.00236	0.00236	0.92539	53	0.00000	0.00856	0.00856	0.16021
16	0.03904	0.00546	0.04450	0.88089	54	0.00000	0.00000	0.00000	0.16021
17	0.00000	0.00937	0.00937	0.87153	55	0.02276	0.00270	0.02546	0.13475
18	0.05970	0.00000	0.05970	0.81182	56	0.00000	0.00543	0.00543	0.12932
19	0.00000	0.00360	0.00360	0.80822	57	0.00000	0.00000	0.00000	0.12932
20	0.00000	0.00775	0.00775	0.80048	58	0.01954	0.00185	0.02139	0.10795
21	0.04973	0.01245	0.06218	0.73830	59	0.00000	0.00387	0.00387	0.10408
22	0.00000	0.00000	0.00000	0.73830	60	0.00000	0.00586	0.00586	0.09819
23	0.00000	0.00452	0.00452	0.73377	61	0.01668	0.00000	0.01668	0.08151
24	0.04511	0.00943	0.05454	0.67923	62	0.00000	0.00177	0.00177	0.07974
25	0.00000	0.00000	0.00000	0.67923	63	0.01983	0.00352	0.02335	0.05640
26	0.00000	0.00372	0.00372	0.67551	64	0.00000	0.00513	0.00513	0.05127
27	0.04157	0.00799	0.04956	0.62595	65	0.00000	0.00000	0.00000	0.05127
28	0.00000	0.01272	0.01272	0.61323	66	0.01330	0.00146	0.01476	0.03652
29	0.00000	0.00000	0.00000	0.61323	67	0.00000	0.00275	0.00275	0.03377
30	0.03858	0.00451	0.04309	0.57014	68	0.00000	0.00376	0.00376	0.03001
31	0.00000	0.00932	0.00932	0.56083	69	0.00955	0.00000	0.00955	0.02046
32	0.05321	0.00000	0.05321	0.50762	70	0.00000	0.00094	0.00094	0.01952
33	0.00000	0.00359	0.00359	0.50403	71	0.00000	0.00174	0.00174	0.01778
34	0.00000	0.00767	0.00767	0.49636	72	0.00641	0.00212	0.00857	0.00921
35	0.04175	0.01211	0.05386	0.44250	73	0.00000	0.00000	0.00000	0.00921
36	0.00000	0.00000	0.00000	0.44250	74	0.00000	0.00051	0.00051	0.00870
37	0.00000	0.00419	0.00419	0.43831	75	0.00374	0.00083	0.00457	0.00413
38	0.03673	0.00862	0.04536	0.39296	76	0.00000	0.00099	0.00099	0.00314
39	0.00000	0.00000	0.00000	0.39296	77	0.00000	0.00000	0.00000	0.00314
40	0.00000	0.00324	0.00324	0.38972	78	0.00174	0.00017	0.00191	0.00123
41	0.03321	0.00691	0.04012	0.34959	79	0.00000	0.00028	0.00028	0.00095
42	0.00000	0.01086	0.01086	0.33874	80	0.00000	0.00023	0.00023	0.00072
43	0.00000	0.00000	0.00000	0.33874	81	0.00054	0.00018	0.00072	0.00000

Table 3.4
Properties of the Two Decision Test Example

TRUE D	P(H0)	R(H1)	ASN
20	0.997053	0.002947	21.4806
25	0.962327	0.037673	28.4823
30	0.738209	0.261791	35.7704
35	0.315228	0.684772	35.8717
40	0.084706	0.915294	29.7936
45	0.019415	0.980585	23.6276

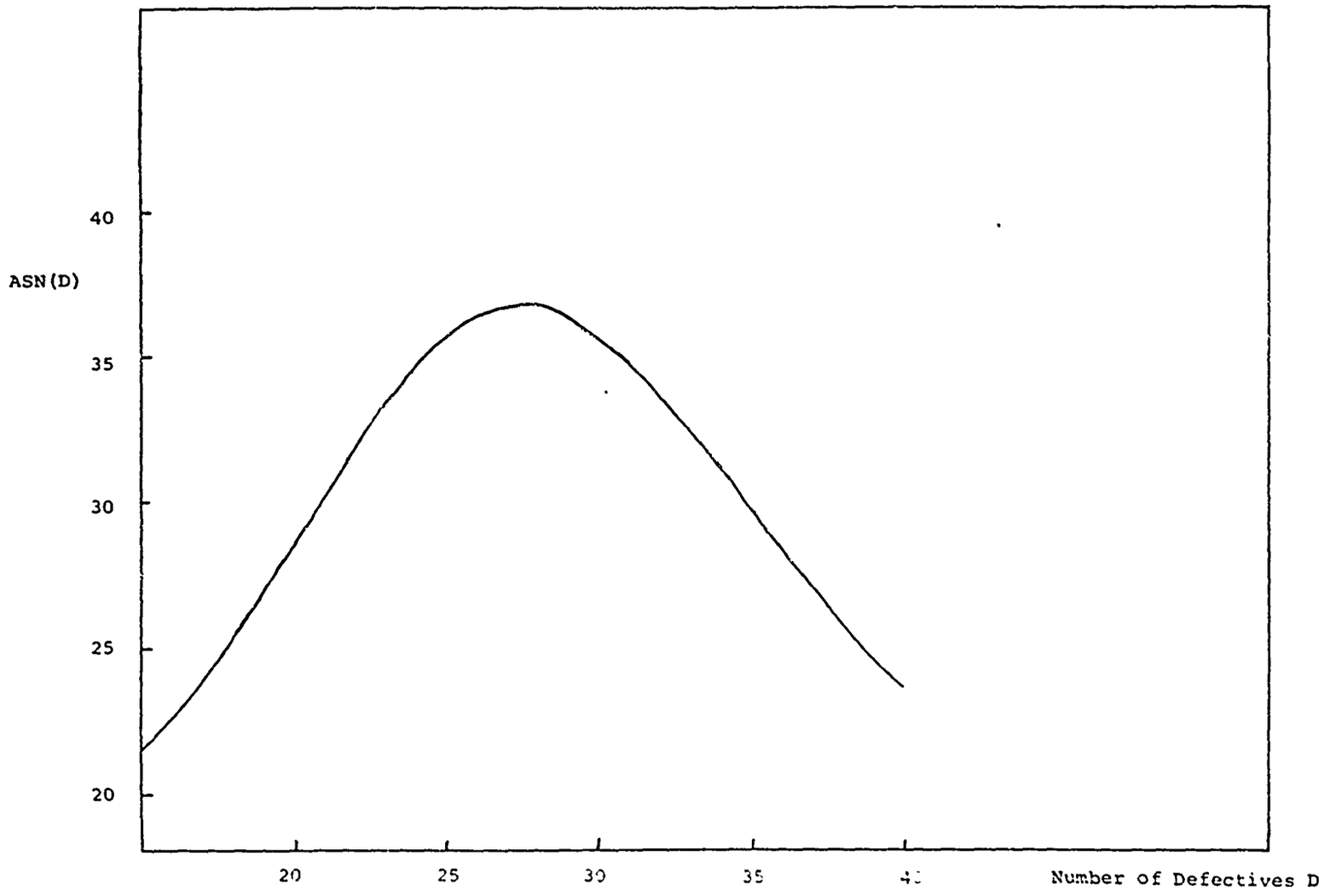

Figure 3.7 Graph of the ASN Function for the Two Decision Example

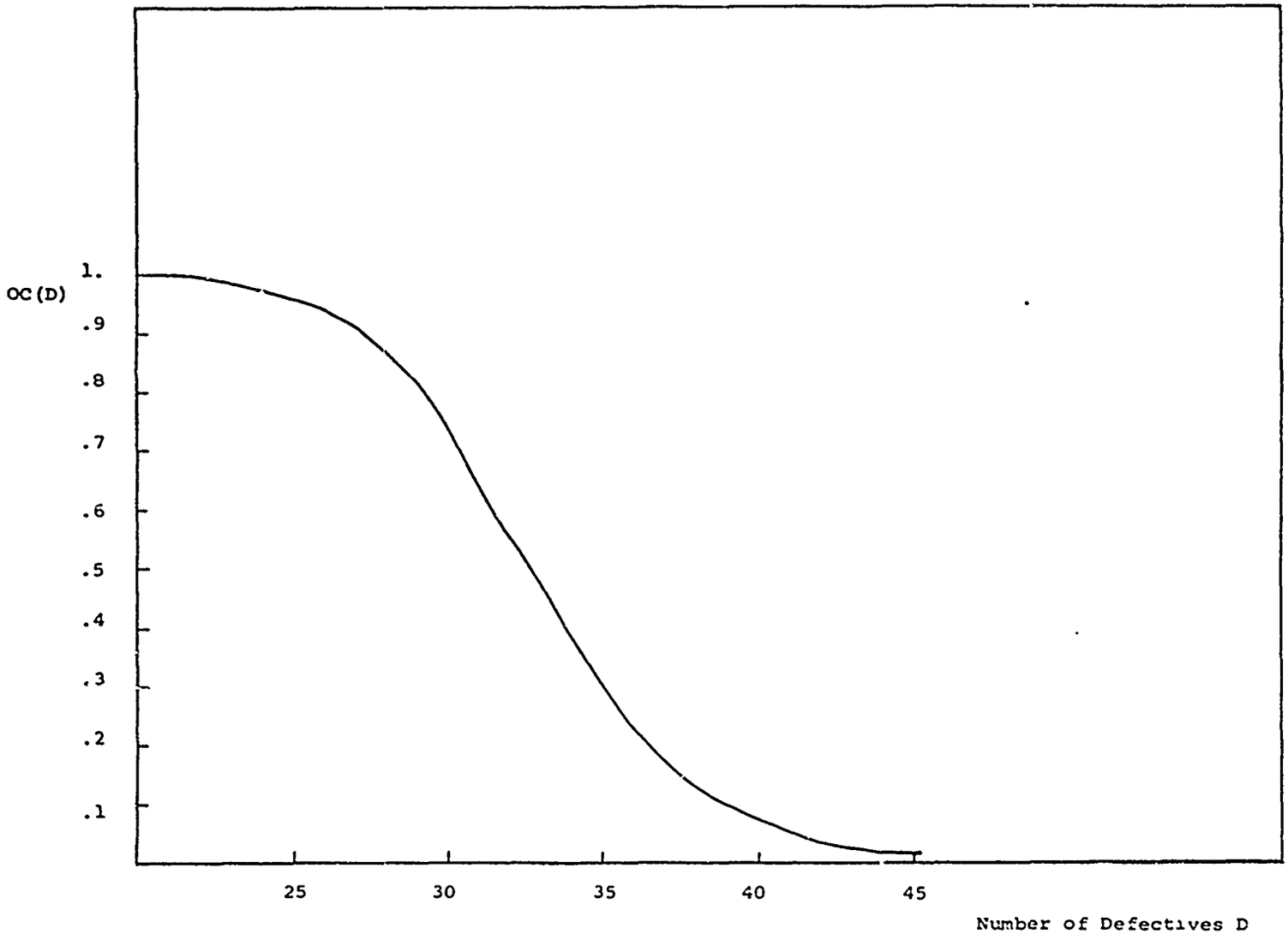

Figure 3.8 Graph of the OC Function for the Two Decision Example

3.4 OBTAINING THE TEST PROPERTIES OF A THREE DECISION TEST REGION

This section will explain how the direct method of sequential analysis is used to find the exact test properties for a three decision sequential test for the hypergeometric distribution. The procedure is easily extended to tests with k>3 decisions. The same test characteristics found for the test in Section 3.3 are calculated for the three decision sequential test region developed in Section 2.2.

The direct method, as applied here, is very similar to the method used for the two decision test treated in Section 3.3. Now, however, there are four possible events at each trial. The events are $A1_n$, $A0_n$, $A2_n$ and C_n, denoting acceptance of H_1, H_0, H_2 at trial n and continuing to trial n+1, respectively. As with the two decision test, it is necessary to compute the probability of each of these events for each trial $n=1,2,\ldots,n_0$, where n_0 is either the truncation point or that point past which the probability of continuation is small enough to ignore.

The probabilities of each of the above events are computed in a manner similar to that used for the two decision test region. One must again compute the probability of reaching each point in the sample space. These probabilities differ from the actual hypergeometric distribution only because of the difference in the number of paths available to reach a given point. The difference in the number of paths is due to the stopping rules of the particular sequential test.

Figure 3.9 shows in detail the grid for part of a typical region for a three decision test. The probability of reaching each point is again computed recursively, moving from one trial to the next. When a point in the continuation region is reached, another sample is taken, illustrated by two arrows leaving each of these points. When a critical point is reached, a decision is made in favor of one of the three hypotheses and the test is terminated. The recursive formula used in calculating the probabilities is:

$$P(x,D,n+1,N)=I(x,n)P(x;D,n,N)((N-n-D+x)/(N-n)) \quad (3.13)$$
$$+I(x-1,n)P(x-1;D,n,N)((D-x+1)/(N-n))$$

where
$$P(x,D,0,N) = \begin{cases} 1 & \text{if } X=0 \\ 0 & \text{otherwise} \end{cases}$$

$$I(x,n) = \begin{cases} 1 & \text{if } (n,x) \in C_n \\ 0 & \text{otherwise} \end{cases}$$

The indicator function I again accounts for the fact that the test ends when one of the critical points is reached.

After all of these probabilities have been computed, the probabilities of the events $A1_n$, $A0_n$, $A2_n$ at each trial are then the probabilities of reaching each of the boundary points. That is,

$$P(Ai_n) = \sum_x J_i(x,n)P(x;D,n,N) \quad (3.14)$$

where
$$J_i(x,n) = \begin{cases} 1 & \text{if } (x,n) \in Ai_n \\ 0 & \text{otherwise} \end{cases}$$

is an indicator function used to sum the probabilities of the proper hypotheses.

Once these probabilities have been found, we can compute the desired test properties as follows. The distribution of the DSN (as a function of D, the true state of nature) is

$$\begin{aligned} P(n;D) &= P(A1_n \cup A0_n \cup A2_n;D) \\ &= P(A1_n;D)+P(A0_n;D)+P(A2_n;D) \end{aligned} \tag{3.15}$$

In a three decision test, there are really three OC functions, each giving the probability of acceptance for one of the hypotheses. The OC function for hypothesis i is found as follows:

$$OC_i(D) = \sum_{n=1}^{n_0} P(Ai_n;D) \tag{3.16}$$

The ASN and VSN of the test (as a function of D) are then computed as

$$ASN(D) = \sum_{n=1}^{n_0} nP(n;D) \tag{3.17}$$

$$VSN(D) = \sum_{n=1}^{n_0} (n-ASN(D))^2 P(n;D) \tag{3.18}$$

The expression for the ASN in (3.9) is also applicable here. The computer program listed in the Appendix calculates the above properties and they have been computed for the sequential test obtained in Section 2.2 and truncated as in Section 3.2 of this chapter. The truncated test region is shown in Table 2.4. The test properties are shown in Table 3.5. Graphs of the ASN and OC functions are shown in Figures 3.10 and 3.11.

Table 3.5
Test Properties of the Three Decision Example

TRUE D	PROB ACCEPT H1	PROB ACCEPT H0	PROB ACCEPT H?	ASN
5	1	0	0	19.6361
6	.99975	.00025	0	20.9057
7	.99846	.00154	0	22.3259
8	.99449	.00551	0	23.8976
9	.98518	.01482	0	25.5987
10	.96671	.03329	0	27.3763
11	.93439	.0656	0	29.1422
12	.88418	.11581	.00001	30.7836
13	.81477	.1852	.00002	32.1885
14	.7288	.27114	.00006	33.2748
15	.63218	.3677	.00012	34.0095
16	.53231	.46745	.00024	34.4121
17	.43612	.56341	.00048	34.5475
18	.34877	.65033	.00091	34.5022
19	.2732	.72514	.00166	34.3732
20	.21036	.78668	.00296	34.2516
21	.15973	.83513	.00515	34.2135
22	.11995	.87129	.00876	34.3163
23	.08932	.89606	.01462	34.5965
24	.06614	.90993	.02392	35.0687
25	.049	.91261	.03839	35.7237
26	.0368	.90293	.06027	36.5276
27	.0287	.87909	.09221	37.4199
28	.02387	.83944	.13669	38.3169
29	.02134	.7834	.19527	39.1193
30	.02001	.7123	.26769	39.7258
31	.01887	.62959	.35154	40.0498
32	.01721	.54039	.4424	40.0336
33	.01482	.45044	.53474	39.6565
34	.0119	.36503	.62307	38.9347
35	.00886	.28815	.70299	37.9143
36	.00612	.22211	.77177	36.6598
37	.00394	.16761	.82845	35.2424
38	.00238	.12415	.87346	33.7296
39	.00137	.09048	.90815	32.1799
40	.00077	.06499	.93424	30.6393
41	.00044	.04607	.95349	29.141
42	.00026	.03226	.96748	27.7077
43	.00016	.02233	.97751	26.3525
44	.00011	.01527	.98462	25.0821
45	.00007	.01033	.9896	23.8981

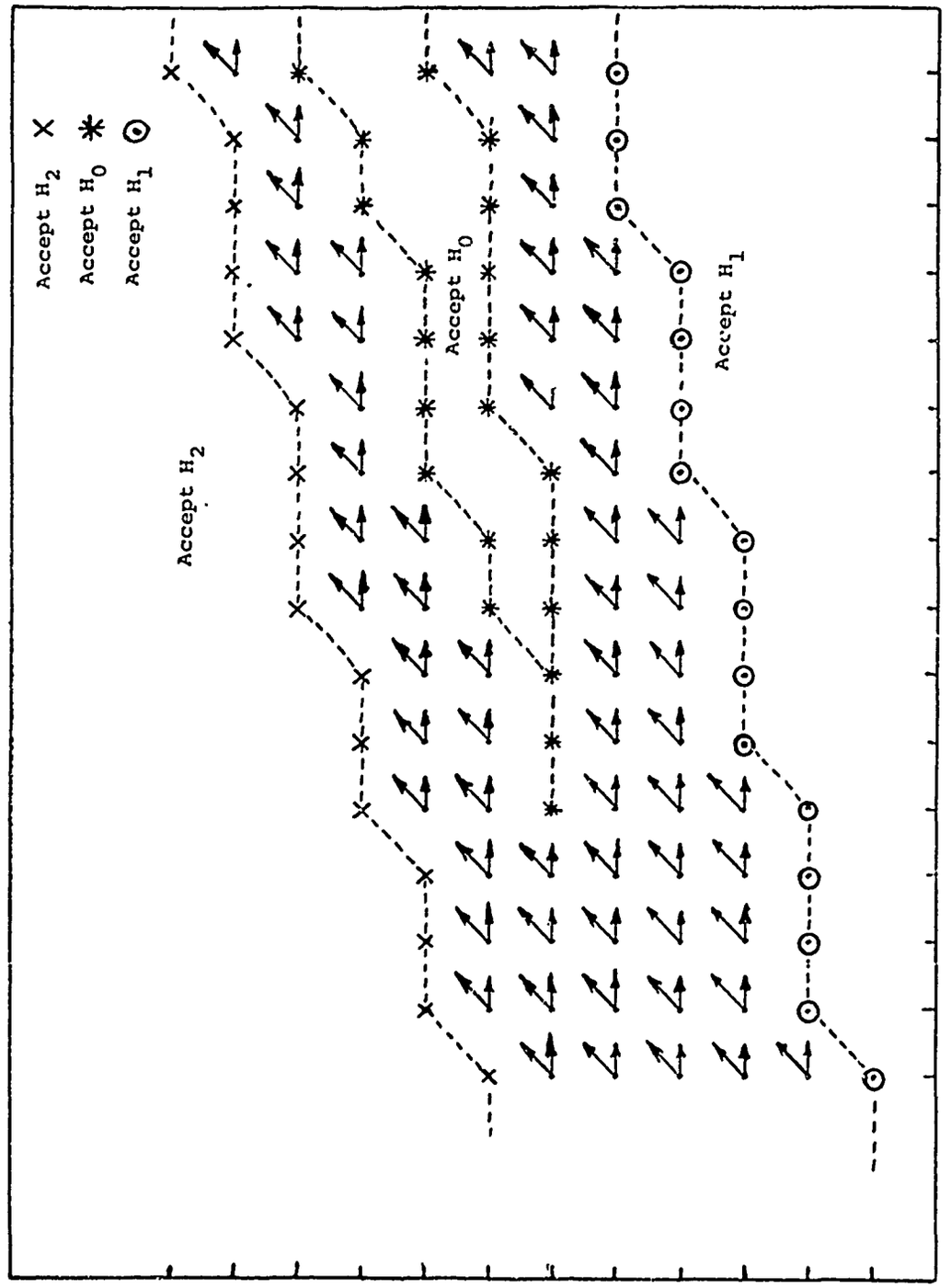

Figure 3.9 Probability Grid for the Three Decision Example

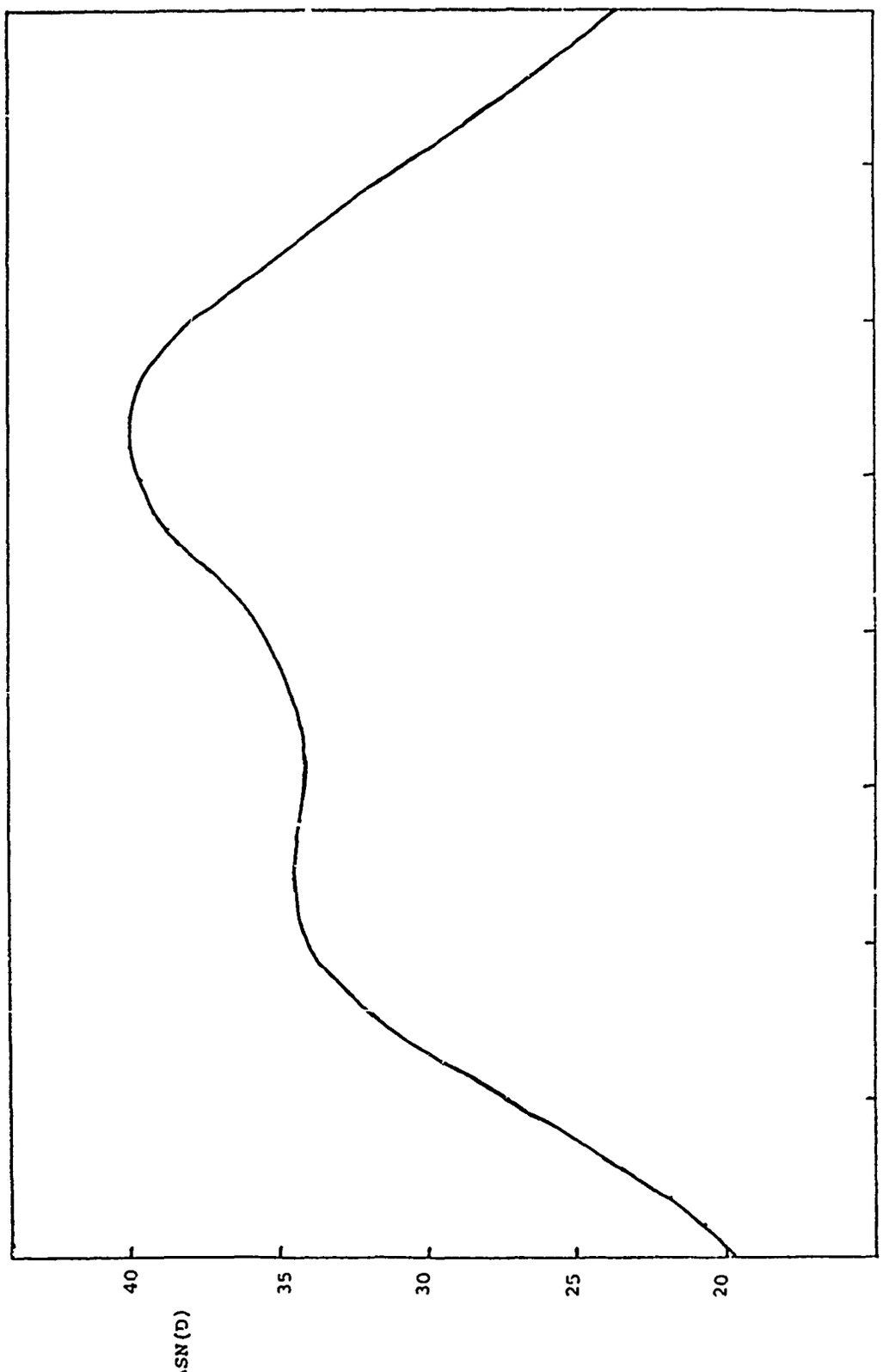

Figure 3.10 Graph of the ASN Function for the Three Decision Example

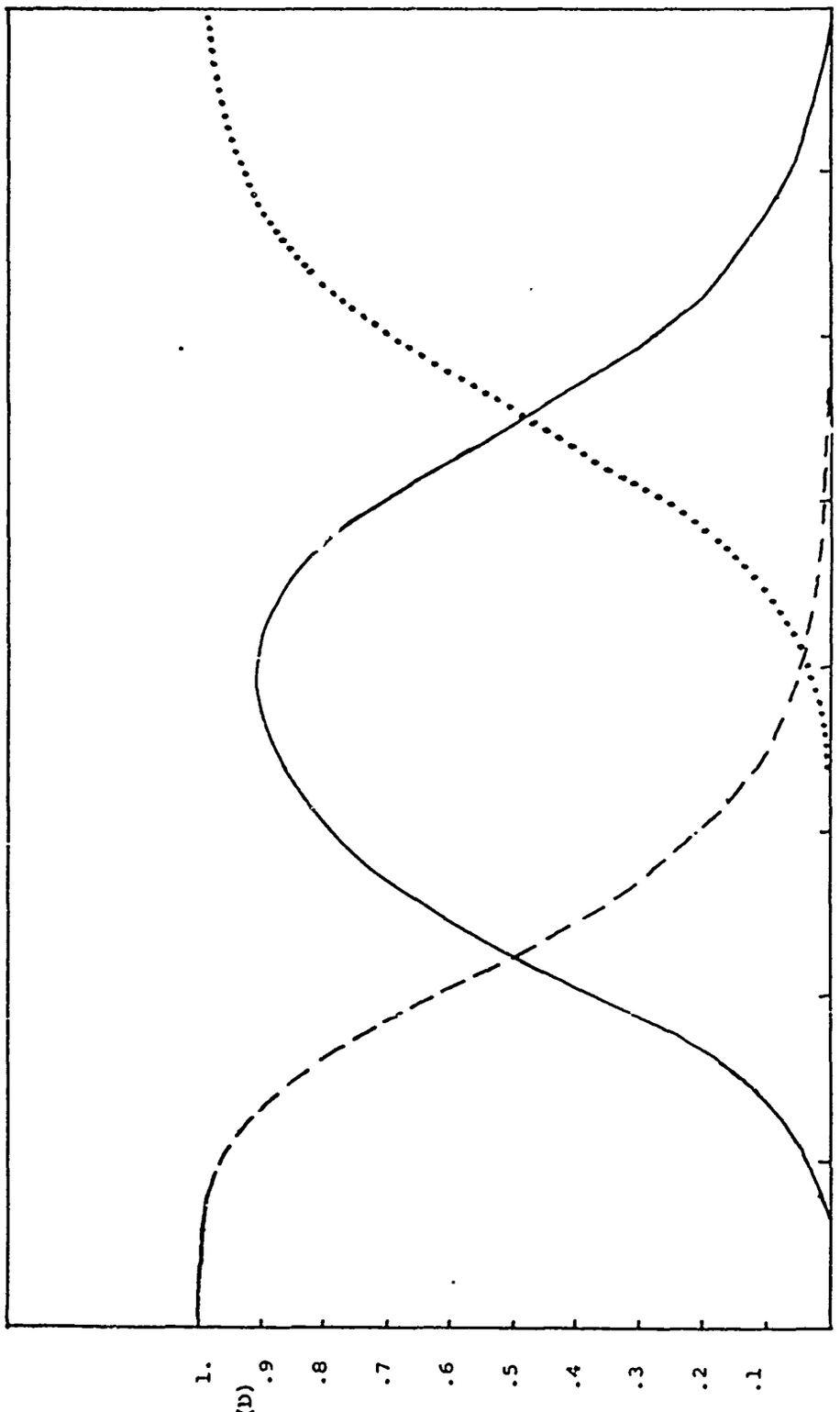

Figure 3.11 Graph of the OC Function for the Three Decision Example

CHAPTER 4

FURTHER NUMERICAL EXAMPLES

4.0 INTRODUCTION

In the first section of this chapter, further examples of two decision truncated sequential tests for the hypergeometric distribution are given. Ten example test plans have been chosen and completely evaluated. These examples have been included to show how the tests perform over a range of different parameter values. The computer program which is listed in the Appendix was used to develop the test regions and to obtain properties of the tests which are given here. This computer program can be used to develop and evaluate tests of the hypergeometric distribution for any desired parameter values. In the second section of this chapter, an illustration is given to show how these sequential tests are used in practice.

4.1 TEST PLAN EXAMPLES

All of the examples of the sequential test plans which are given here are truncated at n*, the sample size of the fixed size sample test. The desired error probabilities are chosen to be α=0.05 and β=0.10. While these error probability limits are not met in all cases when using the sequential test, the true values are always very close to the desired values and could have been made closer following the region modification procedure discussed in Section 3.2.

Because of space limitations, the distribution of the DSN is given only for the first example. For this example and the others which follow, the fixed size test and its corresponding OC function are given. Also, tables of the OC and ASN functions are given for each of the sequential tests. The original sequential test region and the modifications made to it are also shown.

In Chapter 5, using the information given here, comparisons will be made between the sequential tests and both the fixed size sample tests and tests which use the binomial distribution as an approximation. There, the differences among the OC functions of the different tests are examined and the relative advantages of using a sequential procedure is shown. The graphs presented here will aid the comparisons. For each of the test plans, three OC functions have been graphed. These are the OC functions for the fixed size test, for the sequential hypergeometric distribution test, and for the binomial approximation to the hypergeometric distribution test. The ASN function for the sequential hypergeometric distribution tests is also graphed. The horizontal dotted line on the graphs represents the sample size of the fixed size tests.

4.2 USING THE SEQUENTIAL TEST PLANS

This section will present an example which shows how the sequential test plans are used in practice. The example used here could be used, for example, in an acceptance sampling scheme to examine lots of automobile tires when they arrive from the

Test Plan 1

N= 30
D0= 5
D1= 15
ALPHA= 0.050
BETA= 0.100

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 13
CRITICAL VALUE = 4
ALPHA* = 0.00903
BETA* = 0.06971

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
0	1.000000	0.000000
5	0.990969	0.009031
10	0.553923	0.446077
15	0.069710	0.930290
20	0.000414	0.999586
25	0.000000	1.000000

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	*
3	*	3
4	*	3
5	0	4
6	0	4
7	1	4
8	1	4
9	1	5
10	2	5
11	2	5
12	2	5
13	2	3

REGION CHANGE 12 3 5
REGION CHANGE 13 4 5

NUMBER OF DEFECTIVES = 0

TRIAL	P(H0)	P(H1)	P(T)	P(C)
5	1.00000	0.00000	1.00000	0.00000
6	0.00000	0.00000	0.00000	0.00000
7	0.00000	0.00000	0.00000	0.00000
8	0.00000	0.00000	0.00000	0.00000
9	0.00000	0.00000	0.00000	0.00000
10	0.00000	0.00000	0.00000	0.00000
11	0.00000	0.00000	0.00000	0.00000
12	0.00000	0.00000	0.00000	0.00000
13	0.00000	0.00000	0.00000	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
0	1.000000	0.000000	5.0000	0000

NUMBER OF DEFECTIVES = 5

TRIAL	P(H0)	P(H1)	P(T)	P(C)
3	0.00000	0.00246	0.00246	0.99754
4	0.00000	0.00684	0.00684	0.99070
5	0.37283	0.00000	0.37283	0.61787
6	0.00000	0.00151	0.00151	0.61636
7	0.31069	0.00258	0.31327	0.30309
8	0.00000	0.00479	0.00479	0.29860
9	0.00000	0.00000	0.00000	0.29880
10	0.15999	0.00036	0.16035	0.13845
11	0.00000	0.00086	0.00086	0.13759
12	0.09770	0.00147	0.09920	0.03840
13	0.03627	0.00213	0.03840	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
5	0.977475	0.022525	7.4386	6.7248

NUMBER OF DEFECTIVES = 10

TRIAL	P(H0)	P(H1)	P(T)	P(C)
3	0.00000	0.02956	0.02956	0.97044
4	0.00000	0.06568	0.06568	0.90476
5	0.10880	0.00000	0.10880	0.79597
6	0.00000	0.07698	0.07698	0.71899
7	0.13599	0.00376	0.13975	0.57924
8	0.00000	0.07698	0.07698	0.50226
9	0.00000	0.00000	0.00000	0.50226
10	0.08385	0.02634	0.11017	0.39208
11	0.00000	0.04722	0.04722	0.34486
12	0.09639	0.06074	0.15713	0.18773
13	0.12522	0.06261	0.18783	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
10	0.550254	0.449746	8.9493	10.2407

NUMBER OF DEFECTIVES = 15

TRIAL	P(H0)	P(H1)	P(T)	P(C)
3	0.00000	0.11207	0.11207	0.88793
4	0.00000	0.18678	0.18678	0.70115
5	0.02107	0.00000	0.02107	0.68008
6	0.00000	0.09655	0.09655	0.58352
7	0.02634	0.13946	0.16580	0.41772
8	0.00000	0.14098	0.14098	0.27674
9	0.00000	0.00000	0.00000	0.27674
10	0.01000	0.06074	0.07074	0.20600
11	0.00000	0.07266	0.07266	0.13335
12	0.01089	0.06009	0.07098	0.06237
13	0.02425	0.03811	0.06237	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
15	0.092554	0.907446	7.2256	9.5639

NUMBER OF DEFECTIVES = 20

TRIAL	P(H0)	P(H1)	P(T)	P(C)
3	0.00000	0.28079	0.28079	0.71921
4	0.00000	0.31199	0.31199	0.40722
5	0.00177	0.00000	0.00177	0.40546
6	0.00000	0.14687	0.14687	0.25858
7	0.00147	0.13056	0.13203	0.12655
8	0.00000	0.07698	0.07698	0.04957
9	0.00000	0.00000	0.00000	0.04957
10	0.00013	0.02632	0.02644	0.02313
11	0.00000	0.01574	0.01574	0.00739
12	0.00005	0.00578	0.00584	0.00155
13	0.00017	0.00138	0.00155	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
20	0.003595	0.996405	5.0482	4.4136

NUMBER OF DEFECTIVES = 25

TRIAL	P(H0)	P(H1)	P(T)	P(C)
3	0.00000	0.56650	0.56650	0.43350
4	0.00000	0.31472	0.31472	0.11877
5	0.00001	0.00000	0.00001	0.11877
6	0.00000	0.03522	0.03522	0.08355
7	0.00000	0.02841	0.02841	0.00514
8	0.00000	0.00479	0.00479	0.00036
9	0.00000	0.00000	0.00000	0.00036
10	0.00000	0.00036	0.00036	0.00000
11	0.00000	0.00000	0.00000	0.00000
12	0.00000	0.00000	0.00000	0.00000
13	0.00000	0.00000	0.00000	0.00000

TRUE D	P(H0)	R(H1)	ASN	VSN
25	0.000007	0.999993	3.7104	1.1686

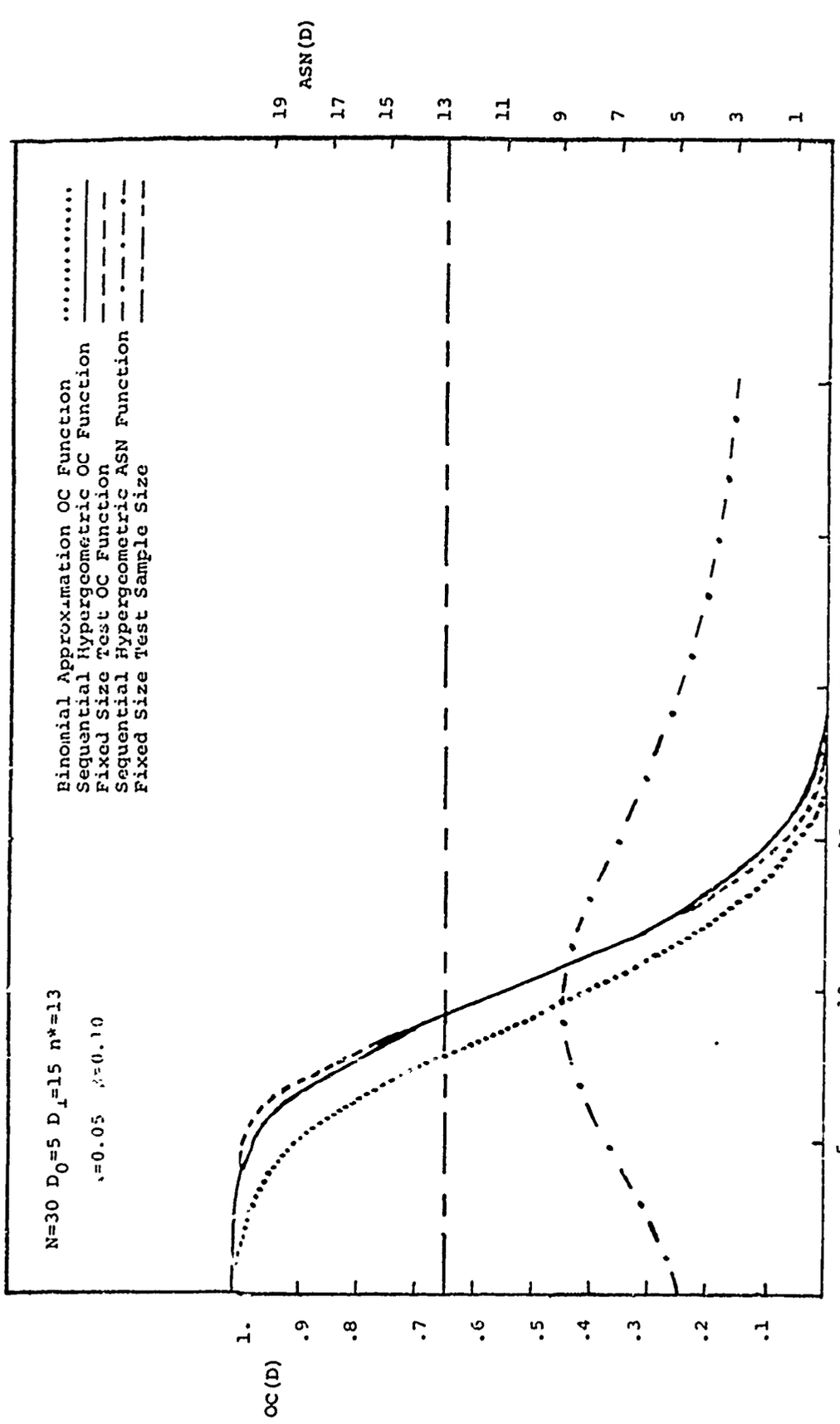

Figure 4.1 Properties for Test Plan 1

Test Plan 2

N= 30
D0= 10
D1= 20
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 13
CRITICAL VALUE = 6
ALPHA* = 0.04508
BETA* = 0.04508

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
6	1.000000	0.000000
8	0.994796	0.005204
10	0.954923	0.045077
12	0.835732	0.164268
14	0.625501	0.374499
16	0.374499	0.625501
18	0.164268	0.835732
20	0.045077	0.954923
22	0.005204	0.994796

TRUNCATE AT THE FIXED SIZE SAMPLE

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	*
3	0	*
4	0	4
5	1	5
6	1	5
7	2	6
8	2	6
9	3	6
10	3	7
11	4	7
12	5	8
13	3	4

REGION CHANGE 9 3 7
REGION CHANGE 12 5 7
REGION CHANGE 13 6 7

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
6	0.999267	0.000733	4.7450	4.8066
8	0.991911	0.008089	5.7193	8.4654
10	0.952488	0.047512	6.8527	11.7979
12	0.844078	0.155922	7.9609	13.5679
14	0.655611	0.344389	8.7860	13.3300
16	0.425591	0.574409	9.1125	12.1022
18	0.220718	0.779282	8.8719	10.8641
20	0.087975	0.912025	8.1679	9.5360
22	0.027026	0.972974	7.2101	7.6405

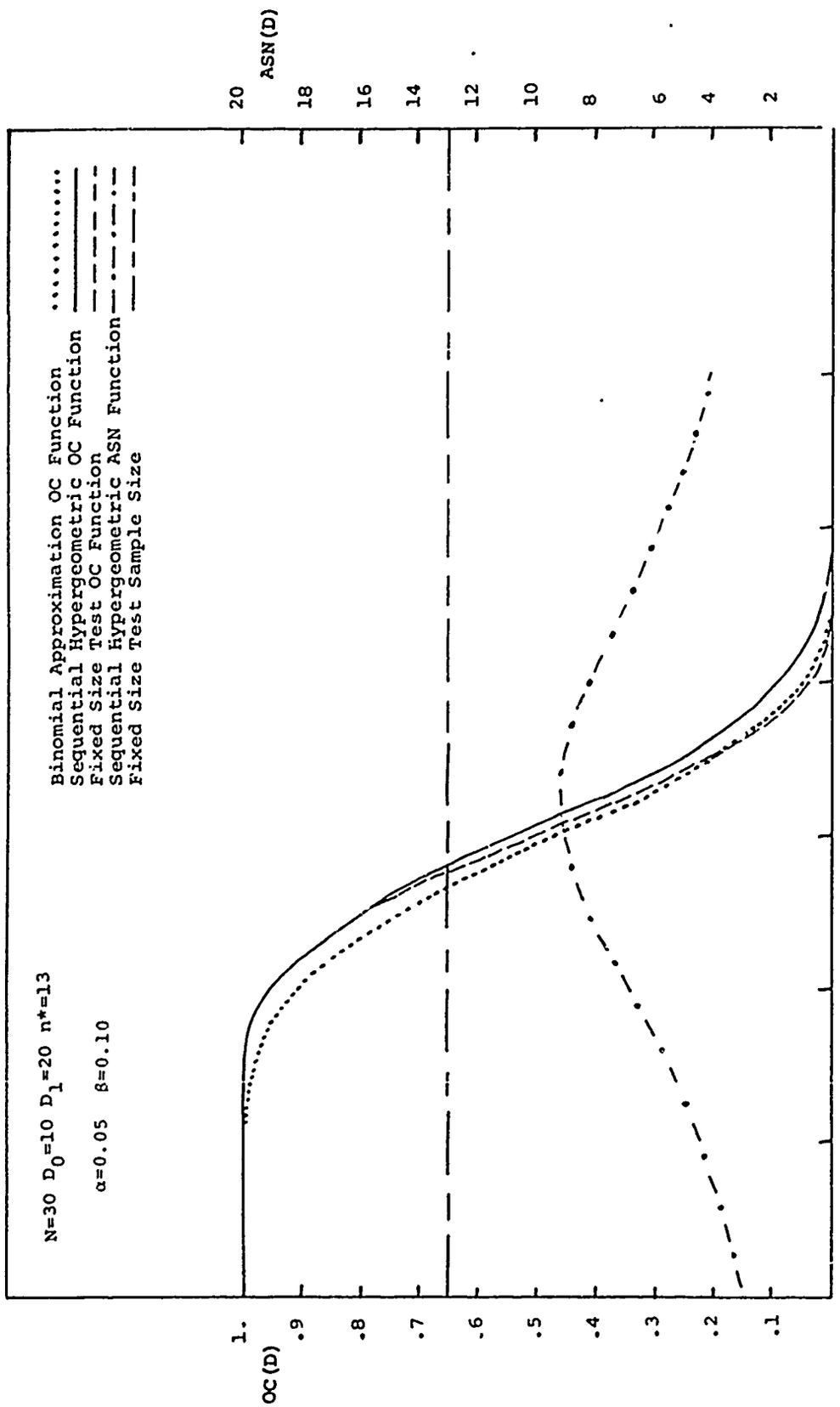

Figure 4.2 Properties for Test Plan 2

Test Plan 3

N= 50
D0= 2
D1= 12
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 19
CRITICAL VALUE = 2
ALPHA* = 0.00000
BETA* = 0.07689

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
0	1.000000	0.000000
2	1.000000	0.000000
4	0.852736	0.147264
6	0.588086	0.411914
8	0.342265	0.657735
10	0.172928	0.827072
12	0.076887	0.923113
14	0.030192	0.969808

TRUNCATE AT THE FIXED SIZE SAMPLE

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	2
3	*	2
4	*	2
5	*	2
6	*	2
7	*	2
8	*	3
9	0	3
10	0	3
11	0	3
12	0	3
13	0	3
14	0	3
15	0	3
16	1	3
17	1	3
18	1	3
19	1	2

REGION CHANGE 8 -1 2
REGION CHANGE 9 -1 2
REGION CHANGE 16 0 3
REGION CHANGE 17 0 3
REGION CHANGE 19 2 3

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	R(H1)	ASN	VSN
0	1.000000	0.000000	10.0000	0.0000
2	0.970612	0.029388	12.6457	16.0443
4	0.791902	0.208098	13.2354	23.0442
6	0.544502	0.455498	12.4740	28.0016
8	0.327993	0.672007	11.1096	29.7424
10	0.178173	0.821827	9.6327	27.5783
12	0.089164	0.910836	8.2877	23.1042
14	0.041841	0.958159	7.1591	18.0613

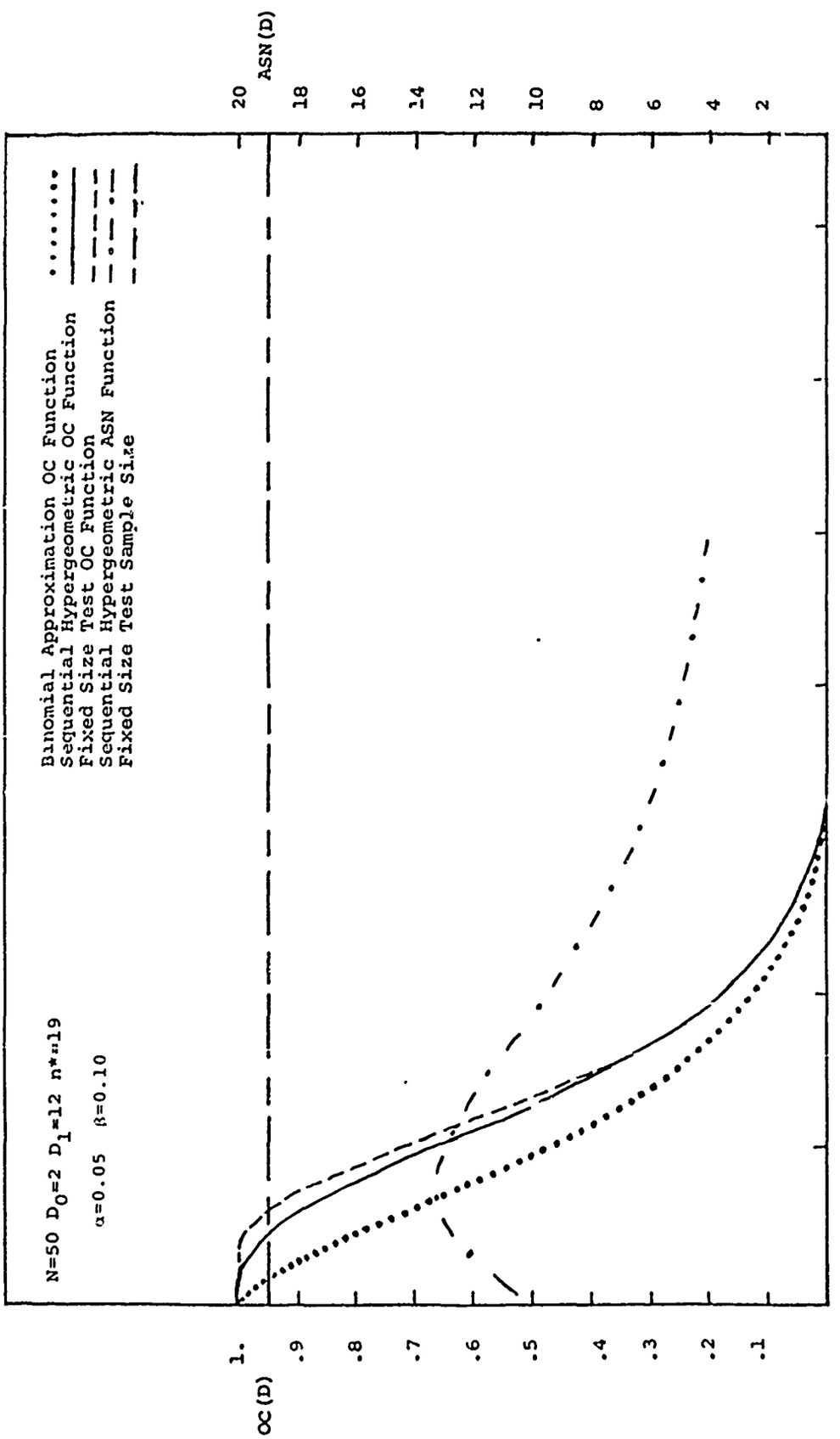

Figure 4.3 Properties for Test Plan 3

Test Plan 4

M= 50
D0= 10
D1= 20
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 25
CRITICAL VALUE = 7
ALPHA* = 0.03688
BETA* = 0.07408

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
6	1.000000	0.000000
8	0.997965	0.002035
10	0.963123	0.036877
12	0.83 554	0.160446
14	0.623193	0.376807
16	0.381207	0.618793
18	0.188577	0.811423
20	0.074080	0.925920
22	0.022502	0.977498
24	0.005061	0.994939

TRUNCATE AT THE FIXED SIZE SAMPLE

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	•	•
2	•	•
3	•	•
4	•	4
5	•	5
6	•	5
7	-	5
8	0	5
9	0	5
10	1	6
11	1	6
12	1	6
13	2	6
14	2	7
15	2	7
16	3	7
17	3	7
18	3	8
19	4	8
20	4	8
21	4	8
22	5	9
23	5	9
24	5	9
25	4	5

REGION CHANGE 13 1 6
REGION CHANGE 16 2 7
REGION CHANGE 19 3 8
REGION CHANGE 22 4 8
REGION CHANGE 23 5 8
REGION CHANGE 24 6 8
REGION CHANGE 25 7 8

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	R(H1)	ASN	VSN
6	0,999547	0,000453	11,5104	15,6764
8	0,993799	0,006201	13,5884	27,2505
10	0,952864	0,047136	15,7954	36,0886
12	0,830905	0,169095	17,5382	38,7901
14	0,627703	0,372297	18,3016	38,3134
16	0,401344	0,598656	17,9546	38,3089
18	0,216236	0,783764	16,7508	38,1897
20	0,099130	0,900870	15,1083	35,7580
22	0,039739	0,960261	13,3937	30,6930
24	0,014660	0,985340	11,8267	24,4193

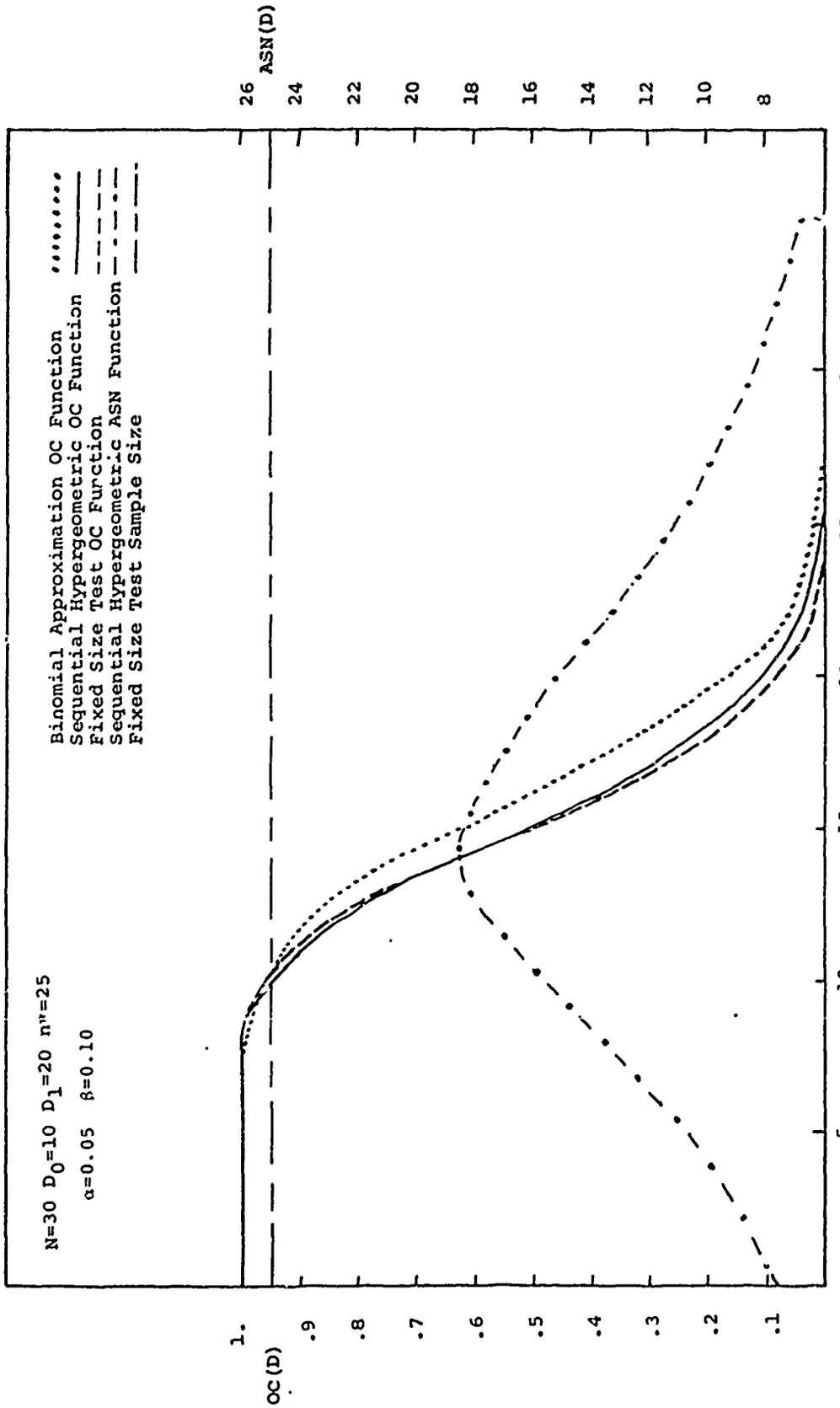

Figure 4.4 Properties for Test Plan 4

Test Plan 5

N= 50
D0= 20
D1= 30
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 28
CRITICAL VALUE = 14
ALPHA= = 0.02641
BETA= = 0.08992

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
16	0.999827	0.000173
18	0.996390	0.003610
20	0.9 588	0.026412
22	0.894892	0.105108
24	0.726855	0.273145
26	0.486568	0.513432
28	0.249650	0.750350
30	0.089923	0.910077
32	0.019865	0.980135
34	0.002056	0.997944

TRUNCATE AT THE FIXED SIZE SAMPLE

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	*
3	*	*
4	*	*
5	*	*
6	0	*
7	1	7
8	1	8
9	2	8
10	2	8
11	3	9
12	3	9
13	4	10
14	4	10
15	5	11
16	6	11
17	6	11
18	7	12
19	7	12
20	8	13
21	8	13
22	9	13
23	9	14
24	10	14
25	11	15
26	11	15
27	12	16
28	7	8

REGION CHANGE 13 3 9
REGION CHANGE 25 11 14
REGION CHANGE 26 12 15
REGION CHANGE 27 13 15
REGION CHANGE 28 14 15

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
16	0,997168	0,002832	12,3499	26,2987
18	0,986983	0,013017	14,2162	35,5592
20	0,950912	0,049088	16,2964	43,5319
22	0,858433	0,141567	18,2756	47,0056
24	0,689213	0,310787	19,6733	45,8024
26	0,465872	0,534128	20,0647	43,1854
28	0,252092	0,747908	19,3452	40,9743
30	0,104829	0,895171	17,8019	37,3319
32	0,033186	0,966914	15,9182	30,7[illegible]4
34	0,008720	0,991280	14,0980	22,8394

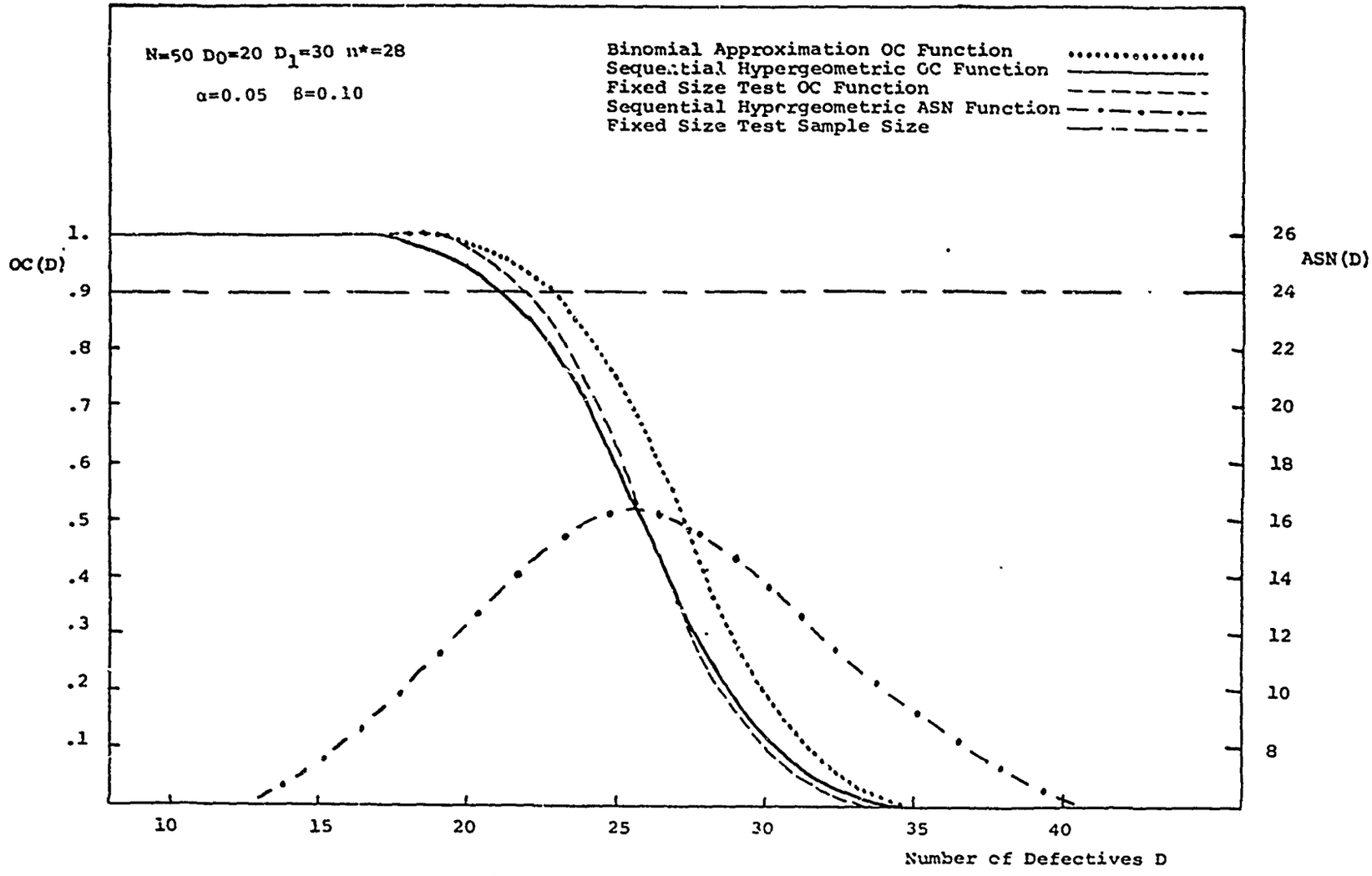

Figure 4.5 Properties for Test Plan 5

Test Plan 6

N= 100
D0= 5
D1= 20
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 29
CRITICAL VALUE = 3
ALPHA* = 0,02398
BETA* = 0,09926

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
0	1,000000	0,000000
5	0,976024	0,023976
10	0,681697	0,318303
15	0,308944	0,691056
20	0,099263	0,900737
25	0,023853	0,976147

TRUNCATE AT THE FIXED SIZE SAMPLE

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	2
3	*	3
4	*	3
5	*	3
6	*	3
7	*	3
8	*	3
9	*	3
10	*	3
11	*	3
12	*	3
13	0	3
14	0	4
15	0	4
16	0	4
17	0	4
18	0	4
19	0	4
20	1	4
21	1	4
22	1	4
23	1	4
24	1	4
25	1	5
26	1	5
27	1	5
28	2	5
29	2	3

REGION CHANGE 13 -1 3
REGION CHANGE 14 -1 4
REGION CHANGE 15 -1 4
REGION CHANGE 16 -1 4
REGION CHANGE 17 0 3
REGION CHANGE 20 0 4
REGION CHANGE 25 1 4
REGION CHANGE 26 1 4
REGION CHANGE 27 1 4
REGION CHANGE 28 2 4
REGION CHANGE 29 3 4

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
0	1,000000	0,000000	17,0000	0,0000
5	0,953815	0,046185	20,9193	21,4321
10	0,641491	0,358509	21,3971	41,4969
15	0,292589	0,707411	18,4645	56,6198
20	0,098854	0,901146	14,8874	52,8936
25	0,026264	0,973736	11,9423	39,7201

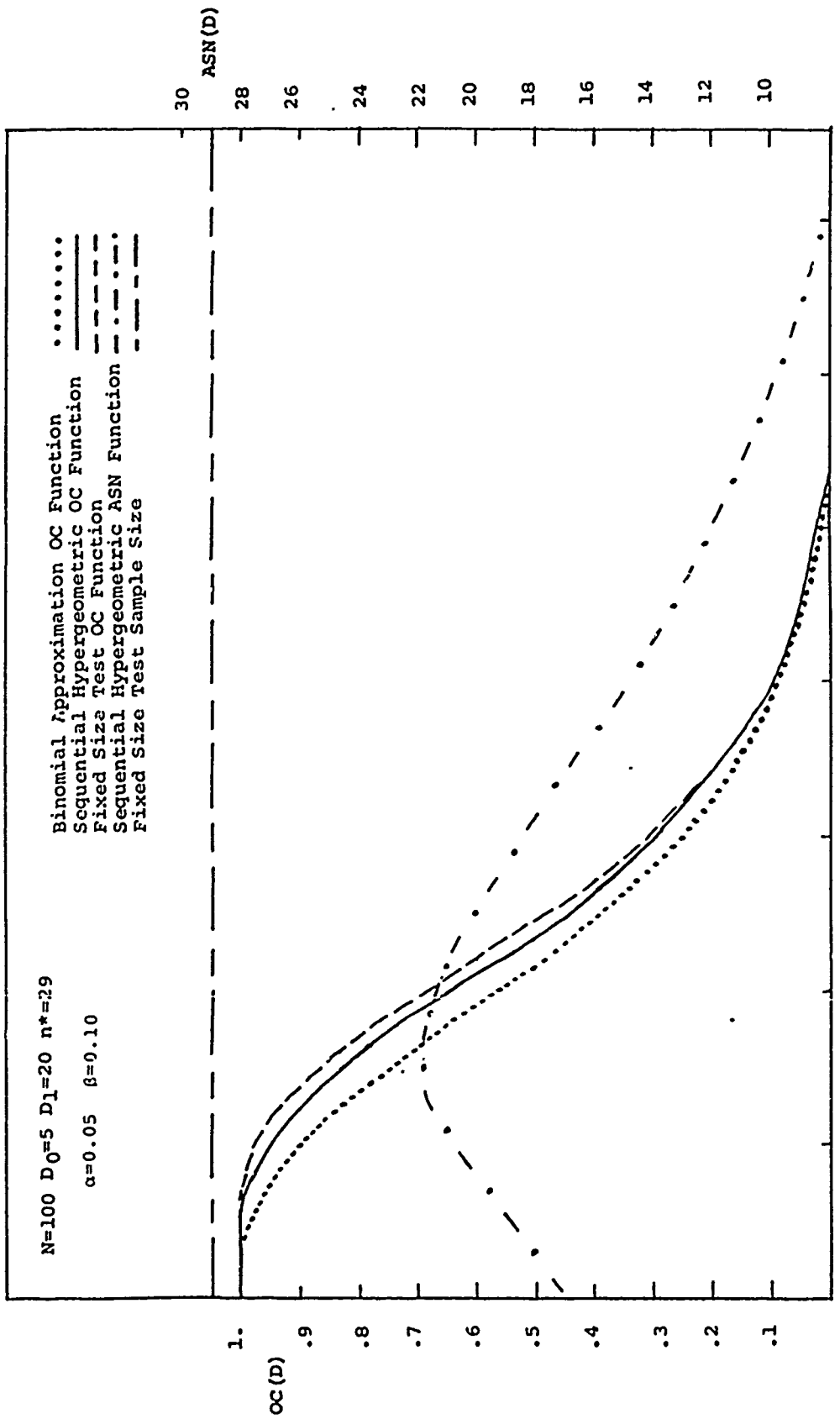

Figure 4.6 Properties for Test Plan 6

Test Plan 7

N= 100
D0= 10
D1= 25
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 37
CRITICAL VALUE = 6
ALPHA* = 0.02845
BETA* = 0.89251

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
0	1.000000	0.000000
5	1.000000	0.000000
10	0.971553	0.028447
15	0.712933	0.287067
20	0.324988	0.675012
25	0.092511	0.907489
30	0.013974	0.986026

TRUNCATE AT THE FIXED SIZE SAMPLE

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
0	1,000000	0,000000	12,0000	0,0000
5	0,999605	0,000395	16,9536	36,0572
10	0,953829	0,046171	23,2090	79,9076
15	0,687567	0,312473	26,1517	89,6052
20	0,325840	0,674160	24,0430	95,1189
25	0,106668	0,893332	19,7799	85,1164
30	0,027316	0,972684	15,8284	61,9862

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	•	•
2	•	•
3	•	3
4	•	4
5	•	4
6	•	4
7	•	4
8	•	4
9	•	4
10	•	4
11	•	5
12	0	5
13	0	5
14	0	5
15	0	5
16	0	5
17	1	5
18	1	6
19	1	6
20	1	6
21	1	6
22	2	6
23	2	6
24	2	6
25	2	7
26	2	7
27	2	7
28	3	7
29	3	7
30	3	7
31	3	7
32	3	7
33	4	8
34	4	8
35	4	8
36	4	8
37	3	4

REGION CHANGE 17 0 5
REGION CHANGE 18 0 5
REGION CHANGE 22 1 6
REGION CHANGE 23 1 6
REGION CHANGE 25 2 6
REGION CHANGE 26 2 6
REGION CHANGE 27 2 6
REGION CHANGE 28 2 7
REGION CHANGE 29 2 7
REGION CHANGE 33 3 7
REGION CHANGE 34 4 7
REGION CHANGE 35 4 7
REGION CHANGE 36 5 7
REGION CHANGE 37 6 7

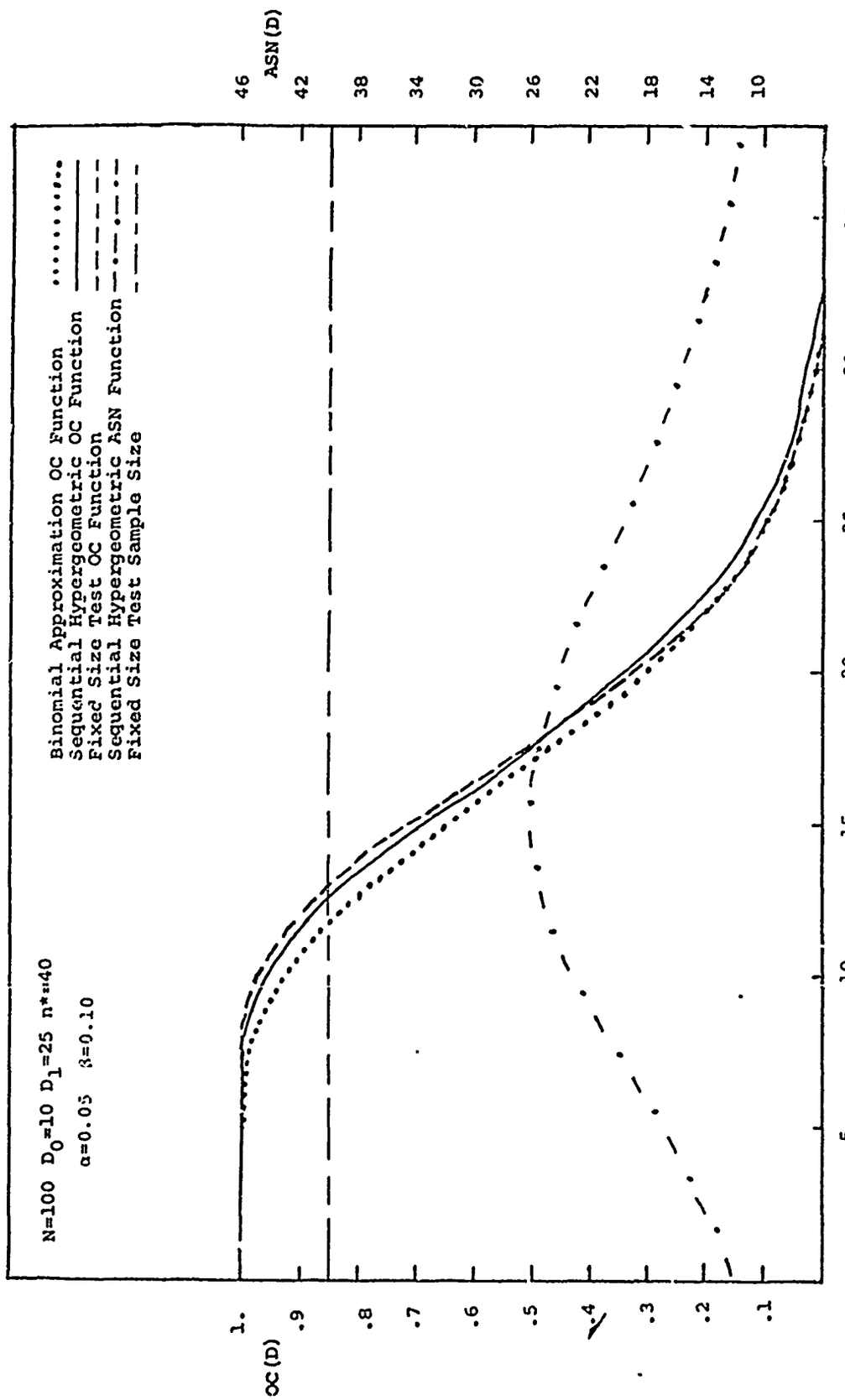

Figure 4.7 Properties for Test Plan 7

Test Plan 8

N= 100
D0= 15
D1= 30
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 42
CRITICAL VALUE = 9
ALPHA* = 0.03531
BETA* = 0.08431

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
10	0.999915	0.000085
15	0.964692	0.035308
20	0.712860	0.287140
25	0.322025	0.677975
30	0.084309	0.915691
35	0.012734	0.987266

TRUNCATE AT THE FIXED SIZE SAMPLE

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
10	0.998707	0.001293	19.5478	51.5726
15	0.953480	0.046520	25.9175	101.2009
20	0.706231	0.293769	30.0855	112.3057
25	0.338401	0.661599	28.8132	117.0773
30	0.104717	0.895283	24.1922	110.5474
35	0.023471	0.976529	19.3708	82.7037

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	*	*
2	*	*
3	*	*
4	*	4
5	*	5
6	*	5
7	*	5
8	*	5
9	*	5
10	*	6
11	0	6
12	0	6
13	0	6
14	0	6
15	1	6
16	1	7
17	1	7
18	1	7
19	2	7
20	2	7
21	2	8
22	2	8
23	3	8
24	3	8
25	3	8
26	3	9
27	4	9
28	4	9
29	4	9
30	4	9
31	4	9
32	5	10
33	5	10
34	5	10
35	5	10
36	6	10
37	6	11
38	6	11
39	6	11
40	7	11
41	7	11
42	5	6

REGION CHANGE 11 -1 6
REGION CHANGE 15 0 6
REGION CHANGE 19 1 7
REGION CHANGE 23 2 8
REGION CHANGE 24 2 8
REGION CHANGE 27 3 9
REGION CHANGE 32 4 10
REGION CHANGE 36 5 10
REGION CHANGE 37 6 10
REGION CHANGE 38 6 10
REGION CHANGE 39 6 10
REGION CHANGE 40 7 10
REGION CHANGE 42 9 10
REGION CHANGE 41 8 10

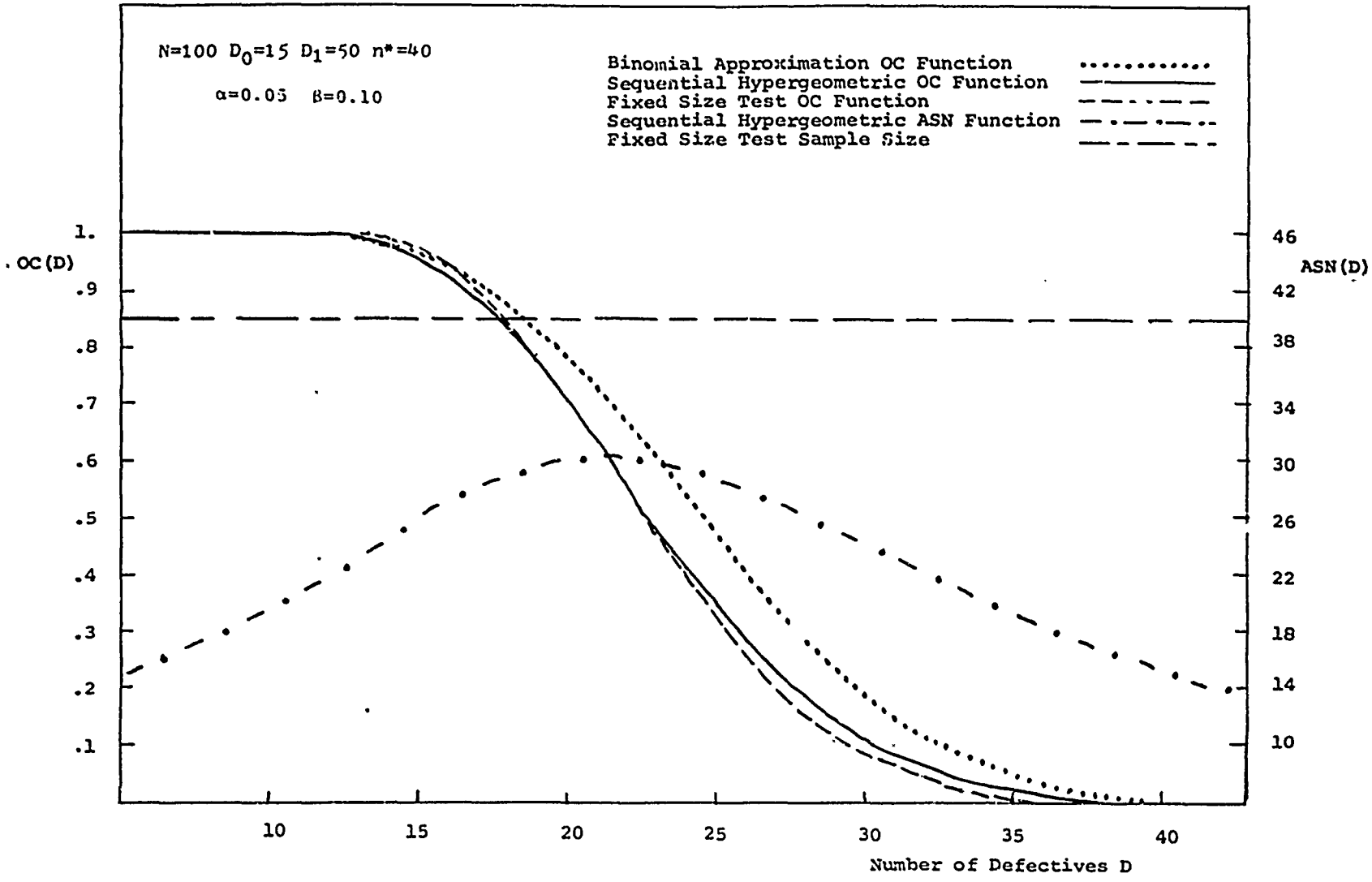

Figure 4.8 Properties for Test Plan 8

Test Plan 9

N= 100
D0= 25
D1= 40
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 50
CRITICAL VALUE = 16
ALPHA= = 0.03171
BETA= = 0.07633

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
20	0.999596	0.000404
25	0.968289	0.031711
30	0.743453	0.256547
35	0.337654	0.662346
40	0.076326	0.923674
45	0.007712	0.992288

TRUNCATE AT TRIAL 50

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
20	0.996533	0.003467	21.8661	79.4777
25	0.951746	0.048254	28.4359	142.2807
30	0.729948	0.270052	33.8801	165.0477
35	0.358360	0.641640	33.8326	161.9724
40	0.104832	0.899168	28.9978	144.6135
45	0.020934	0.979066	23.3165	101.8754

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	•	•
2	•	•
3	•	•
4	•	•
5	•	•
6	•	6
7	•	7
8	•	7
9	•	7
10	0	7
11	0	8
12	0	8
13	1	8
14	1	9
15	1	9
16	2	9
17	2	9
18	3	10
19	3	10
20	3	10
21	4	10
22	4	11
23	4	11
24	5	11
25	5	12
26	5	12
27	6	12
28	6	12
29	6	13
30	7	13
31	7	13
32	8	14
33	8	14
34	8	14
35	9	14
36	9	15
37	9	15
38	10	15
39	10	16
40	10	16
41	11	16
42	11	16
43	11	17
44	12	17
45	12	17
46	13	17
47	13	18
48	13	18
49	14	18
50	9	10

REGION CHANGE 18 2 9
REGION CHANGE 32 7 14
REGION CHANGE 44 11 17
REGION CHANGE 45 11 17
REGION CHANGE 46 12 17
REGION CHANGE 47 13 17
REGION CHANGE 48 14 17
REGION CHANGE 49 15 17
REGION CHANGE 50 16 17

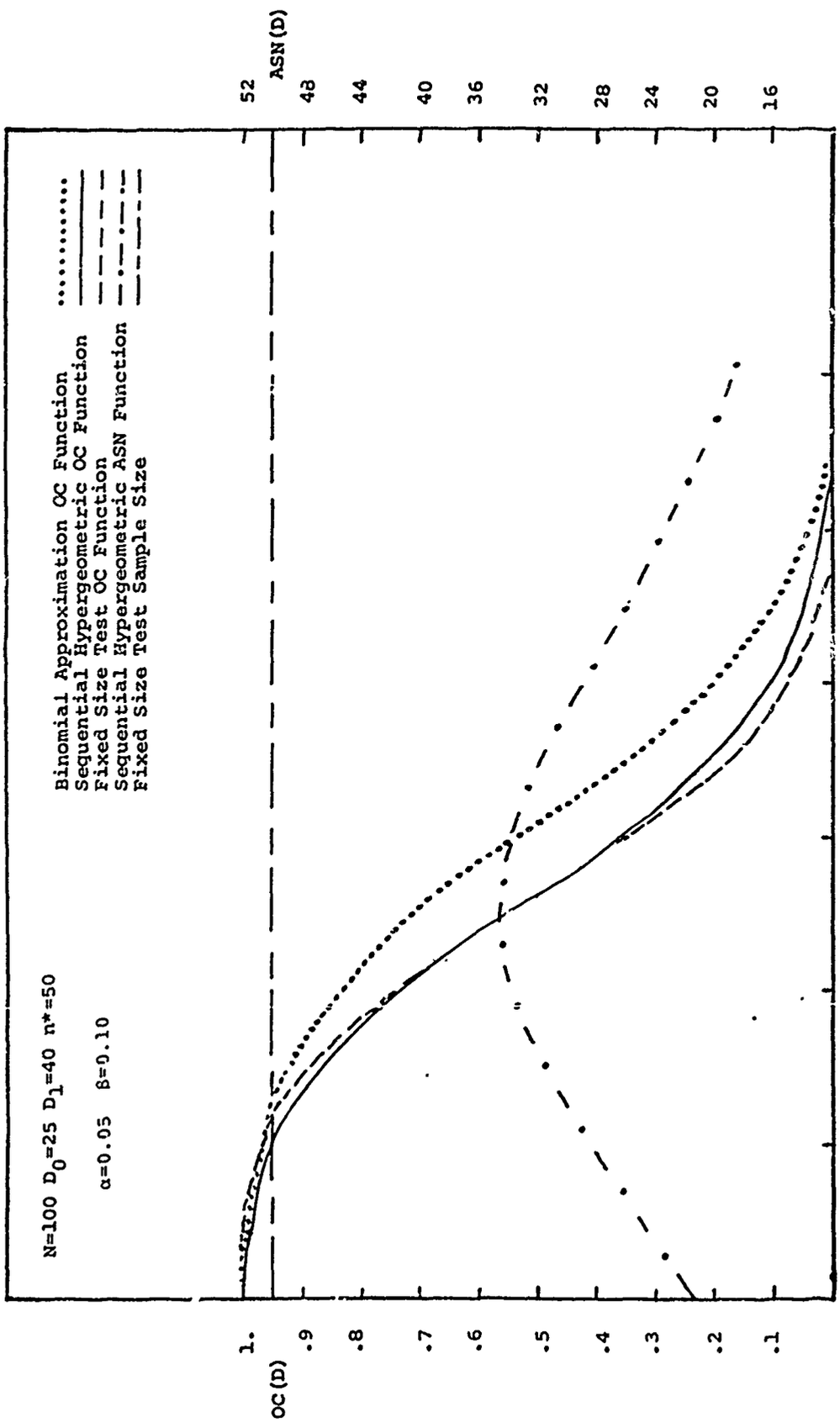

Figure 4.9 Properties for Test Plan 9

Test Plan 10

N= 100
D0= 40
D1= 60
ALPHA= 0.050
BETA= 0.100
EVALUATION OF REGION REQUESTED

THE FIXED SIZE TEST IS AS FOLLOWS:
SAMPLE SIZE = 40
CRITICAL VALUE = 20
ALPHA* = 0,03053
BETA* = 0,07257

FIXED SIZE TEST OC FUNCTION:

D	ACCEPT H0	ACCEPT H1
40	0,969468	0,030532
42	0,936909	0,063091
44	0,883426	0,116574
46	0,805132	0,194868
48	0,702338	0,297662
50	0,580792	0,419208
52	0,451057	0,548943
54	0,325989	0,674011
56	0,217219	0,782781
58	0,132135	0,867865
60	0,072568	0,927432

TRUNCATE AT THE FIXED SIZE SAMPLE

SEQUENTIAL TEST PROPERTIES

TRUE D	P(H0)	P(H1)	ASN	VSN
40	0,953206	0,046794	20,4036	89,2391
42	0,916610	0,083390	22,0433	99,2028
44	0,861244	0,138756	23,6398	105,9496
46	0,784629	0,215371	24,7794	109,1421
48	0,687702	0,312298	26,0252	109,3923
50	0,575568	0,424432	26,6403	107,9166
52	0,456909	0,543091	26,7605	105,8276
54	0,342060	0,657940	26,3779	103,5056
56	0,240412	0,759588	25,5432	100,4747
58	0,158179	0,841821	24,3532	95,8481
60	0,097378	0,902622	22,9295	88,9986

THE WALD REGION IS AS FOLLOWS:

TRIAL	ACCEPT H0	ACCEPT H1
1	•	•
2	•	•
3	•	•
4	•	•
5	•	•
6	0	•
7	0	7
8	1	8
9	1	8
10	2	9
11	3	9
12	3	10
13	4	10
14	4	11
15	5	11
16	5	12
17	6	12
18	6	12
19	7	13
20	7	13
21	8	14
22	8	14
23	9	15
24	9	15
25	10	16
26	10	16
27	11	17
28	12	17
29	12	18
30	13	18
31	13	18
32	14	19
33	14	19
34	15	20
35	15	20
36	16	21
37	16	21
38	17	22
39	17	22
40	11	12

REGION CHANGE 6 -1 -1
REGION CHANGE 28 11 17
REGION CHANGE 11 2 9
REGION CHANGE 13 3 10
REGION CHANGE 40 60 20
REGION CHANGE 37 17 21
REGION CHANGE 38 18 21
REGION CHANGE 39 19 21
REGION CHANGE 40 20 21

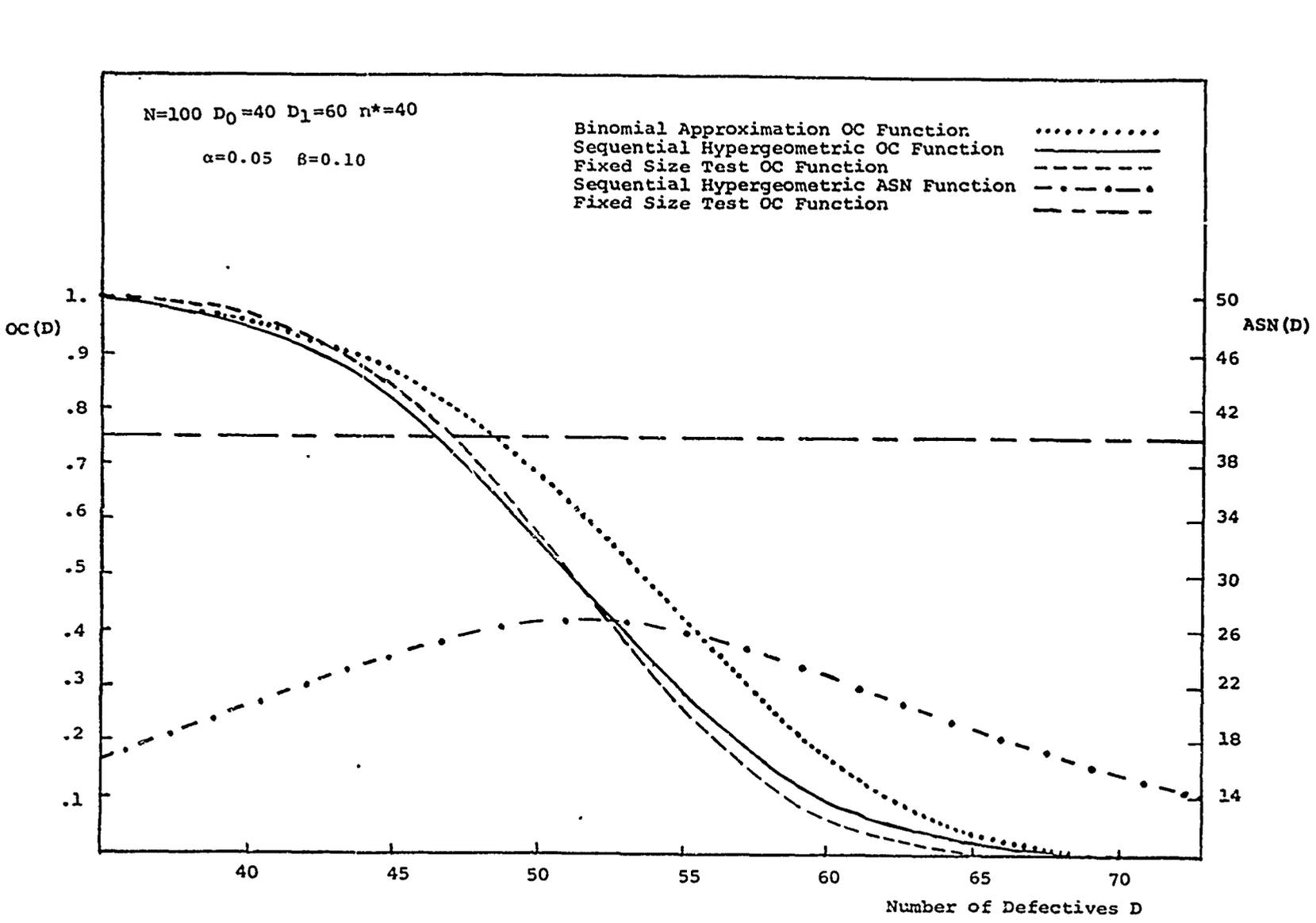

Figure 4.10 Properties for Test Plan 10

manufacturer. Each lot contains 30 tires and it is desired to accept the lot if it contains $D \leq 5$ defectives ($D_0=5$) and to reject the lot if it contains $D \geq 15$ defectives ($D_1=15$). The α and β errors are chosen to be 0.05 and 0.10 respectively. For this test, Test Plan 1 is appropriate. The truncated sequential test region and the test properties are given in Section 4.1.

To carry out the sequential test, the following procedure is followed. At each stage of the test a tire is selected at random (without replacement) from those remaining in the lot. The total number of defectives which have been observed is then compared with the critical limits which define the test region. This is continued until one of the critical limits is reached.

A typical sequential sample which might be obtained for this case is shown in Table 4.1, along with the critical limits. This is shown graphically in Figure 4.11. For this particular sample, only 1 defective has been found at the 7th trial. A decision is therefore made in favor of H_0 and the lot is accepted.

Table 4.1
Typical Sequential Sample

Trial	$c_L(n)$	x	$c_U(n)$
1	*	0	*
2	*	0	*
3	*	1	3
4	*	1	3
5	0	1	4
6	0	1	4
7	1	1	4
8	1		4
9	1		5
10	2		5
11	2		5
12	3		5
13	4		5

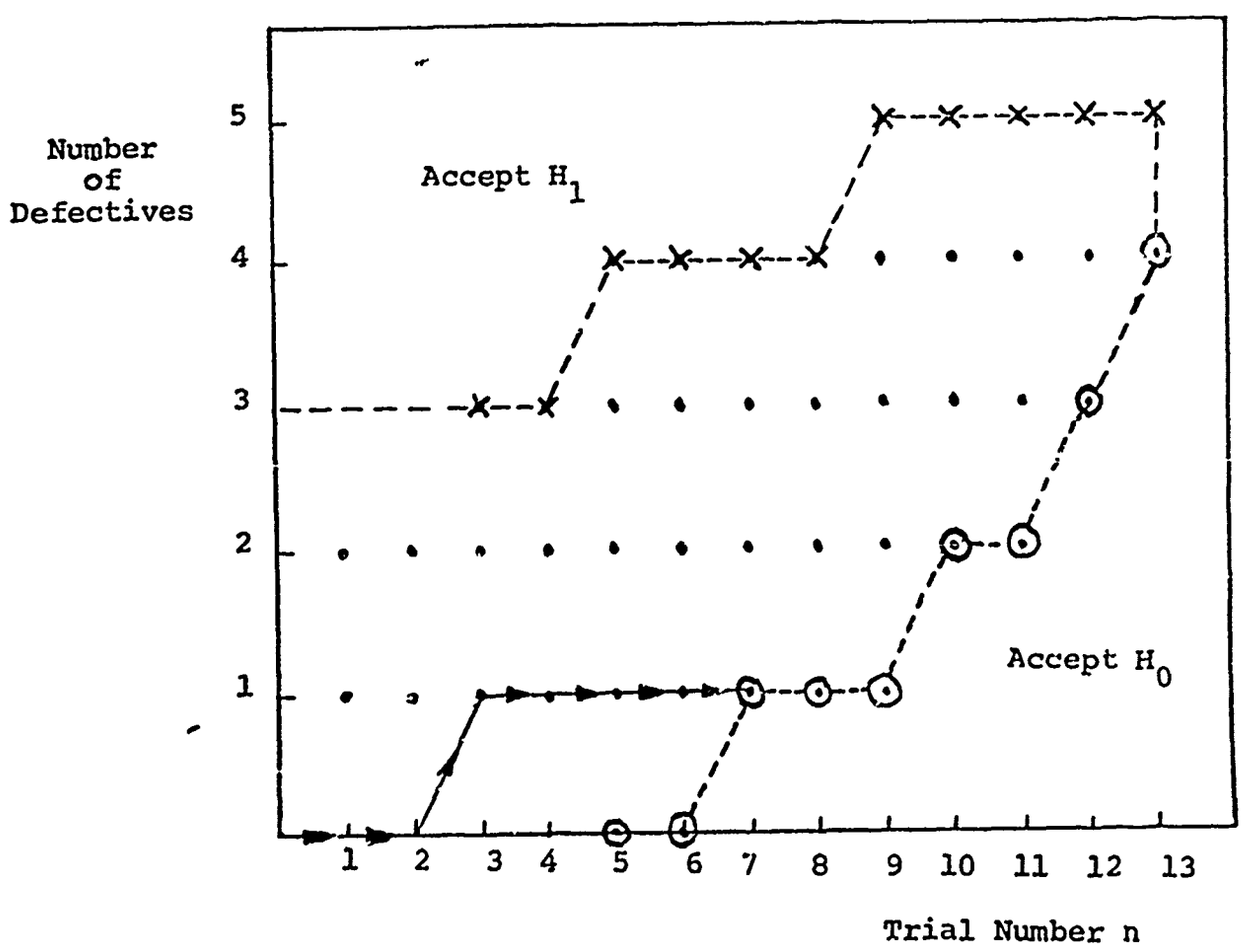

Figure 4.11 Graphical View of the Sequential Test

CHAPTER 5

COMPARISON WITH OTHER TESTS

5.0 INTRODUCTION

In this chapter, two types of comparisons are presented. In the first section, comparisons between the fixed size sample and the sequential tests of the hypergeometric distribution are given. In the second section, sequential tests for the hypergeometric distribution are compared with similar tests for the binomial distribution. The purpose of the first section is to show the relative superiority (with respect to the expected sample size) of the truncated sequential test when compared to fixed size sample tests for the same hypothesis. The second section shows the possible consequences of using Wald type regions for the binomial distribution to develop approximate sequential tests for the hypergeometric distribution.

5.1 COMPARISON WITH FIXED SIZE SAMPLE TESTS

In this section the fixed size tests described in Section 1.3 will be compared and contrasted with the sequential tests given in Section 2.1 and truncated as in Section 3.2. There will be frequent references to the tables and graphs given in Chapter 4. The OC functions for these two tests will be examined along with the ASN function of the truncated sequential test. From this, it will be quite easy to see the relative gain in efficiency (with respect to the expected sample size) obtained by using the sequential test.

The fixed size tests used here were developed as described in Section 1.3. They are the (non-randomized) tests with minimum sample size such that the limits on the desired error probabilities are met. From examination of the OC function for these fixed size tests (these OC functions and their graphs are given in Section 4.1), it is seen that these error probabilities are usually quite a bit smaller than the desired error probabilities. This will have to be taken into consideration (in a subjective manner) when making the comparisons presented here.

The graphs for each example test plan in Section 4.1 show the OC functions for both the fixed size sample test (dashed line) and the sequential test (solid line), as well as for the binomial approximation test (dotted line), which is explained in the next section.

Table 5.1 gives the true α and β errors for each of the tests, along with the sample size for the fixed size test (which is also the truncation point for the sequential test) and the maximum value of the ASN function for the sequential case. The following observations can be made about these example test plans.

The most important advantage gained by using a sequential test procedure is an overall reduction in the amount of sampling required to come to a decision. This advantage is again demonstrated here. The maximum of the ASN function for each of the test plans ranges between 25 and 33 percent below the sample size of the fixed size test. There is no doubt that these tests will result in a considerable saving with respect to sampling costs. There is, however, a small price to be paid for this saving.

Table 5.1
Comparison of the Fixed Size
and Sequential Tests

				Fixed Size Test			Sequential Test		
Plan #	N	D_0	D_1	α^*	β^*	n^*	α'	β'	$\max_{0 \le D \le N} \{ASN\}$
1	30	5	15	0.0093	0.0697	13	0.0225	0.0925	8.94
2	30	10	20	0.0450	0.0451	13	0.0475	0.0879	9.11
3	50	2	12	0.0000	0.0769	19	0.0294	0.0892	13.24
4	50	10	20	0.0369	0.0741	25	0.0471	0.0991	18.30
5	50	20	30	0.0264	0.0899	28	0.0491	0.1048	20.05
6	100	5	20	0.0240	0.0993	29	0.0462	0.0989	21.78
7	100	10	25	0.0284	0.0925	37	0.0462	0.1067	26.15
8	100	15	30	0.0353	0.0843	42	0.0465	0.1044	30.25
9	100	25	40	0.0317	0.0763	50	0.0483	0.0777	34.62
10	100	40	60	0.0305	0.0726	40	0.0468	0.0974	26.76

The α and β error probabilities of the fixed size tests are never any larger, and are usually somewhat smaller, than the desired error limits. The α and β errors for the sequential test, while somewhat larger than those of the fixed size test, are in all cases quite close and rarely above the desired limits. Examination of the graphs of the OC function also illustrates this point. From the graphs, it is again seen that the error probabilities have increased somewhat, but that over all, the OC curves are very similar. This is the price paid for the sample size advantage described in the last paragraph.

It should be pointed out that randomized tests meeting the required error probability limits, which may have slightly smaller sample sizes than do the fixed size tests presented here, can usually be found. This type of test is not often used in practice and tends to be somewhat of a theoretical contrivance. Even when compared to such randomized tests, the sequential tests are usually superior.

5.2 COMPARISON WITH SEQUENTIAL TESTS OF THE BINOMIAL DISTRIBUTION

In this section, the Wald regions for tests of the binomial distribution are examined. First, the physical characteristics of these regions are compared with the regions for the hypergeometric distribution which were found in Section 2.1. Then the binomial regions are used as approximate tests for the hypergeometric distribution.

For completeness, the well-known method of determining the regions for a sequential test of the binomial distribution (Wald, 1947) is briefly reviewed here. The same notation developed in Chapter 2 will be used.

The probability mass function of the binomial distribution is

$$f(x,n,p)=\binom{n}{x}p^{x}(1-p)^{n-x} \qquad 0\leq x\leq n \tag{5.1}$$

This is the limiting distribution of the hypergeometric distribution if N approaches infinity and D/N remains constant. The likelihood ratio used to test the hypothesis

$$\begin{aligned} &H_0\colon\ p=p_0 \\ \text{versus}\quad &H_1\colon\ p=p_1>p_0 \end{aligned} \tag{5.2}$$

is

$$\frac{L(x;p_1,n)}{L(x;p_0,n)} = \frac{p_1^{x}(1-p_1)^{n-x}}{p_0(1-p_0)^{n-x}} \tag{5.3}$$

where x is the number of defects observed after n trials. Following the same procedure given in Chapter 2, the rules for the test are

$$\begin{aligned} &\text{accept } H_0 \text{ if} \quad L(x;p_1,n)/L(x;p_0,n)\leq B \\ &\text{accept } H_1 \text{ if} \quad L(x;p_1,n)/L(x;p_0,n)\geq A \end{aligned} \tag{5.4}$$

and otherwise take another sample. The log likelihood ratio function is found to be

$$g(x,p_0,p_1,n)=\ell n(L(x,p_1,n)/L(x,p_0,n)) \qquad (5.5)$$
$$=x\cdot\ell n(p_1/p_0)+(n-x)\cdot\ell n((1-p_1)/(1-p_0))$$

The critical values for the log likelihood ratio are then

$$b=\ell n(B)=\ell n(\beta/(1-\alpha)) \qquad (5.6)$$
$$a=\ell n(A)=\ell n((1-\beta)/\alpha)$$

We then find the critical values for acceptance and rejection at each trial from the inverse function g^{-1} when solving for x. The equations for these are

$$k_L(n)=\left[g^{-1}(b,p_0,p_1,n)\right] \qquad (5.7)$$
$$=\left[(b-n\ell n((1-p_1)/(1-p_0))/\ell n(p_1(1-p_0)/(p_0(1-p_1)))\right]$$
$$k_U(n)=\left[g^{-1}(a,p_0,p_1,n)\right]+1$$
$$=\left[(a-n\ell n((1-p_1)/(1-p_0))/\ell n(p_1(1-p_1)))\right]+1$$

where K=[R] is again the greatest integer less than or equal to [R].

The inverse functions in (5.7) are linear in n and can therefore be represented by two parallel lines with slope

$$s=((1-p_0)/(1-p_1)/\ell n(p_1(1-p_0)/(p_0(1-p_1))) \qquad (5.8)$$

and intercepts

$$I_L=\ell n(\beta/(1-\alpha))/\ell n(p_1(1-p_0)/(p_0(1-p_1))) \qquad (5.9)$$
$$1_U=\ell n((1-\beta)/\alpha)/\ell n(p_1(1-p_0)/(p_0(1-p_1)))$$

An example of these two lines, which define the boundary of the sequential test regions, is shown in Figure 5.1. In the examples which follow, the critical limits $k_L(n)$ and $k_U(n)$ are computed

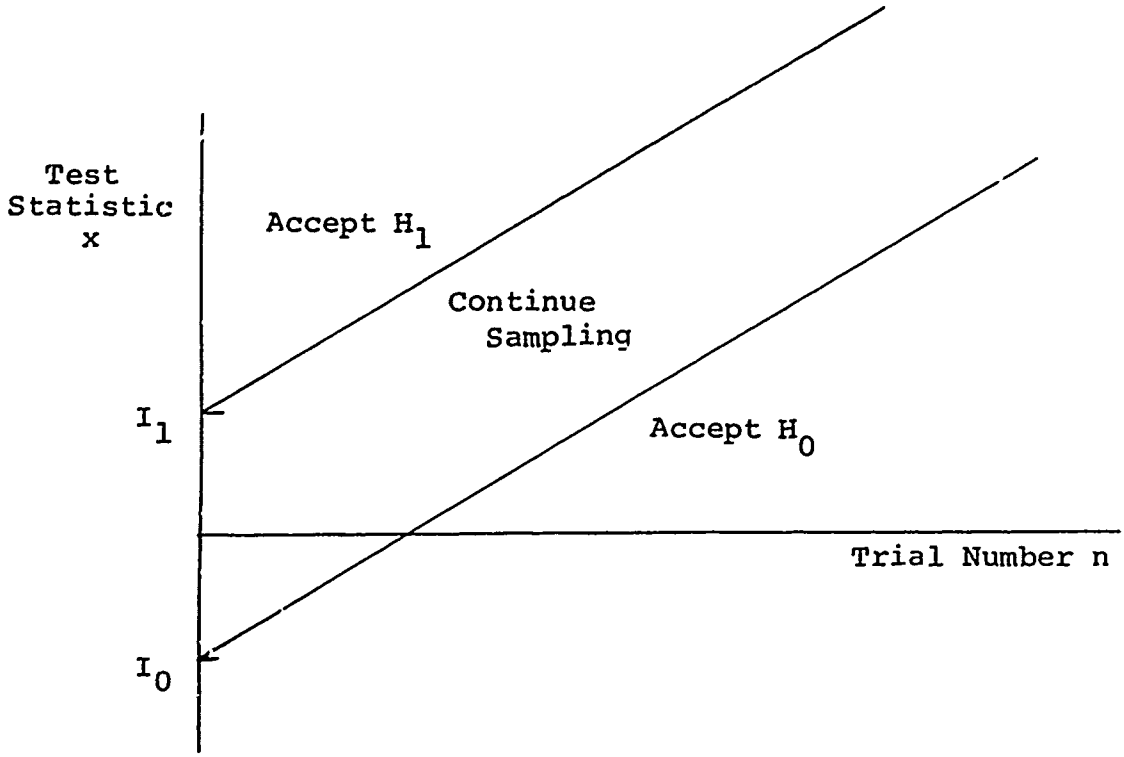

Figure 5.1 Typical Wald Region for the Binomial Distribution

and compared with the critical limits of the sequential tests of the hypergeometric distribution, $c_L(n)$ and $c_U(n)$.

When using the binomial test regions to approximate the test for the hypergeometric distribution, one must make a decision as to which values to use for p_0 and p_1. Some approximations for hypergeometric distribution are given, for example, by Johnson and Kotz (1969). These approximations are dependent on the sample size n and their use in a sequential test would be as complicated as the exact test is. For this reason p_0 and p_1 are taken here to be

$$p_0=D_0/N$$
$$p_1=D_1/N \qquad (5.10)$$

For large N and small n, these are good approximations and would likely be the values used in practice. Equation (5.7) is used to construct the binomial test regions, with the following modifications. First, if D_0+1 defects are observed, the test is terminated and H_1 is accepted, as it is then known that H_0 cannot be true. Also, the test is truncated, again following the procedure given in Section 3.2. No other modifications are made to the test regions.

Table 5.2 gives the critical limits for both the truncated hypergeometric test plan ($c_L(n)$ and $c_U(n)$) and the binomial approximation to this test ($k_L(n)$ and $k_U(n)$) for Test Plan 3 (from Section 4.1) where n=50, D_0=2 and D_1=12. This same information is given for Test Plan 5 in Table 5.3.

Because these two distributions are the same at the first trial, the critical values will be the same for n=1. The main difference between these two tests is that the hypergeometric distribution test regions tend to shrink in width, especially from above. The average distance between $c_L(n)$ and $c_U(n)$ decreases with n until the truncation point is reached, while the average distance between $k_L(n)$ and $k_U(n)$ remains the same except at the points where the test is truncated. The shrinking of the hypergeometric distribution test region is due to the finiteness of the population. Also, the binomial regions usually tend to be shifted by a small amount from the hypergeometric test region. The effects of this are shown in the numerical results to follow.

Table 5.2
Binomial and Hypergeometric Test Regions
for Test Plan #3 (N=50, D_0=2, D_1=12)

Trial	$c_L(n)$	$c_U(n)$	$k_L(n)$	$k_U(n)$
1	*	1	*	1
2	*	2	*	1
3	*	2	*	1
4	*	2	*	1
5	*	2	*	2
6	*	2	*	2
7	*	2	*	2
8	*	2	*	2
9	*	2	*	2
10	0	3	0	2
11	0	3	0	2
12	0	3	0	2
13	0	3	0	2
14	0	3	0	3
15	0	3	0	3
16	0	3	0	3
17	0	3	0	3
18	1	3	1	3
19	2	3	2	3

Table 5.3
Binomial and Hypergeometric Test Regions
for Test Plan #5 (N=50, D_0=20, D_1=30)

Trial	$c_L(n)$	$c_U(n)$	$k_L(n)$	$k_U(n)$
1	*	*	*	*
2	*	*	*	*
3	*	*	*	*
4	*	*	*	*
5	*	*	*	*
6	0	*	0	6
7	1	7	0	7
8	1	8	1	7
9	2	8	1	8
10	2	8	2	8
11	3	9	2	9
12	3	9	3	9
13	3	9	3	10
14	4	10	4	10
15	5	11	4	11
16	6	11	5	11
17	6	11	5	12
18	7	12	6	12
19	7	12	6	13
20	8	13	7	13
21	8	13	8	14
22	9	13	9	14
23	9	14	10	15
24	10	14	11	15
25	11	14	12	16
26	12	15	13	16
27	13	15	14	16
28	14	15	15	16

Table 5.4 summarizes the results of using the binomial regions as an approximation. The actual α and β errors, along with the maximum values of the ASN function, are given for each test plan in Section 4.1.

It is seen from Table 5.4 that in almost all of the examples, either the true α or the true β error is quite a bit larger than what was desired and that error in the opposite direction is smaller than what was desired. This can also be seen in the shifts in the graphs of the OC function for the approximate tests, as shown in Section 4.1. This seems to indicate that the test approximation is, in a sense, biased. There seems to be no significant difference between the maximum values of the ASN function for these tests. This is reasonable, as the tests are truncated at the same trial.

The above observations indicate two things to the user of these sequential tests. First, the binomial regions do not, in general (except for cases where the population size N is large when compared with a typical sample size of the test procedure), provide adequate tests for the hypergeometric distribution. Also, the importance of finding the exact test properties to compare with the desired values is clearly shown.

Table 5.4
Comparison of the Sequential Hypergeometric Test
and the Binomial Approximation

				Binomial Approximation			Hypergeometric Test		
Plan #	N	D_0	D_1	α''	β''	$\max_{0\le D\le N}\{ASN\}$	α'	β'	$\max_{0\le D\le N}\{ASN\}$
1	30	5	15	0.0989	0.0600	7.86	0.0225	0.0925	8.94
2	30	10	20	0.0880	0.0475	9.11	0.0475	0.0879	9.11
3	50	2	12	0.1812	0.0634	10.38	0.0294	0.0892	13.24
4	50	10	20	0.0402	0.1574	18.72	0.0471	0.0991	18.30
5	50	20	30	0.0224	0.1987	21.96	0.0491	0.1048	20.05
6	100	5	20	0.1030	0.0879	18.71	0.0462	0.0989	21.78
7	100	10	25	0.0723	0.0922	23.71	0.0462	0.1067	26.15
8	100	15	30	0.0460	0.1883	29.52	0.0465	0.1044	30.25
9	100	25	40	0.0398	0.2031	37.31	0.0483	0.0777	34.62
10	100	40	60	0.0409	0.1770	26.39	0.0468	0.0974	26.76

CHAPTER 6

ESTIMATION OF THE NUMBER OF DEFECTIVES AFTER TERMINATION OF THE SEQUENTIAL TEST

6.0 INTRODUCTION

In this chapter, a method whereby one can obtain a point estimator and/or confidence intervals for the number of defectives in a finite population is discussed. The estimation is to be performed after sequential tests, such as those discussed in Chapter 2, have been terminated.

In general, there are two basic types of sequential estimation to be considered. First, there is the problem of sampling sequentially until estimates with the desired degree of precision have been obtained. This might be expressed, for example, by specifying the maximum allowable confidence interval length. Another type of estimation is often required when one would like to estimate the parameter in question after completion of a sequential hypothesis test. This latter type of estimation is considered here.

A brief outline of the history of sequential estimation is presented here first. This is followed by a description of a general method of estimation (given by Schmee (1974) and Goss (1974b)) which may be used after a sequential test. The following sections show how this method is applied to sequential tests of the hypergeometric distribution. A numerical example is also given.

6.1 HISTORY OF SEQUENTIAL ESTIMATION

This section presents a brief overview of some of the approaches to sequential estimation which have received attention in the past. More comprehensive overviews of sequential estimation are given, for example, by Johnson (1961), Wetherill (1966), Goss (1974a) and Schmee (1974).

The first results with sequential estimation were obtained by Tweedie (1945) and Haldane (1945) who use an inverse binomial sampling technique. Wald (1947), in his book, gives some structure to the problem and suggests an approach which he admits is not optimal. While the problem was not solved by Wald, his work seems to have led to the later results given by Anscombe (1953) and Cox (1952a and 1952b). These procedures, which deal with generalized boundaries, were also explored by Wolfowitz (1946), Blackwell (1947), Savage (1947) and Knight (1965). Dixon and Mood (1948) apply the "up-down" method to sequential estimation, and Anscombe (1953) reviews the early methods of sequential estimation. Most of these early efforts were primarily aimed at estimation rather than hypothesis testing followed by estimation.

Armitage, in his discussion of Anscombe's paper (Anscombe, 1953), points out that in general, the suggested sequential estimation techniques are not any better than the standard fixed size procedures. He also stresses the need for methods of estimation to be performed after sequential tests of hypotheses. The following is a brief review of the work which has been done with such methods.

Girschick et al. (1946) give a simple method of finding the unique unbiased estimator (UBE) for a binomial SPRT. For tests of the binomial parameter, Armitage (1958) compares the mean square error of the maximum likelihood estimator (which is unbiased in a fixed size test, but not necessarily in a sequential test) with the variance of the UBE and gives a method of finding confidence intervals which meet the classical probability statement, and which are dependent on the stopping rules. Aroian and Oksoy (1972) present a Bayesian procedure for estimation and for finding confidence intervals. The procedure is to be used at the completion of a sequential life test. A generalization of this procedure is due to Schmee (1974) and Goss (1974c) and is the basis of the estimation procedures developed here.

To complete our overview of sequential estimation, the Bayesian approaches should be mentioned. This subject is treated by Wetherill (1966) and DeGroot (1970). Box and Tiao (1973) discuss the sequential nature of Bayesian methods of estimation.

6.2 THE GENERAL METHOD OF SCHMME AND GOSS

The method of estimation presented here is similar in nature to the method given by Aroian and Oksoy (1972). Schmee (1974) and Goss (1974c), in their respective dissertations, give a general method of finding point estimates and confidence intervals which can be easily applied to sequential methods. Schmee (1974) applies the method to sequential tests of the mean of a normal distribution with variance both known and unknown. Goss treats the two-sided

test for the mean of a normal distribution with variance known (Goss,1974b) and tests of the binomial distribution (Goss,1974a). A description of the method follows; it is applied to tests of the hypergeometric distribution in the following sections.

In order to present the method in a general fashion, it is best to consider estimation of a continuous parameter from a continuous distribution (e.g., the mean of the normal distribution). When estimation is to be performed for a discrete parameter or from a discrete population, the appropriate integrals will become summations and probability density functions will become probability mass functions.

When the direct method of sequential estimation is used, one computes the probability of reaching or crossing each point on the region's boundary (assuming a test in discrete time), for different values of the true state of nature. The method for doing this is treated in Chapter 3. Let $P(A_n|\theta)$ and $P(R_n|\theta)$ represent the probability of accepting and rejecting H_0 respectively at trial n if θ is the true state of nature. These probabilities are easily found by using the direct method. If we take a Bayesian view of the situation and assume a uniform prior for θ (over all possible values of θ), a "pseudo-posterior" (in the sense that it is not based on a sufficient statistic) distribution for θ can be found by using Bayes' Formula.

$$P(\theta|A_n) = \frac{P(A_n|\theta)}{\int_\theta P(A_n|\theta)d\theta}$$

$$P(\theta|R_n) = \frac{P(R_n|\theta)}{\int_\theta P(R_n|\theta)d\theta} \qquad (6.1)$$

The first equation is used if the hypothesis is accepted at trial n; the second one is used if the hypothesis is rejected there. Although this method is valid for certain distributions (e.g., the binomial and hypergeometric), there can be both theoretical and practical objections to it in other cases.

The first objection is that if there is more than one outcome contained in the events A_n or R_n, that information might be lost by grouping the outcomes into one event. The second objection concerns the restriction to the uniform or "non-informative" prior distribution. This is especially true with the so-called "improper prior" which ranges between plus and minus infinity and cannot be made to integrate to one (Box and Tiao, 1973). Both of these objections are eliminated with the more general method developed by Schmee and Goss.

The general method makes use of all of the available information and allows the use of any specified prior distribution. Let $P(\theta)$ denote the prior density of θ and let $f(x,n|\theta)$ denote the density of x given θ, where x is the sequential sample obtained up to trial n (where H_0 is either accepted or rejected) and θ is the true state of nature to be estimated. If $T(x)$ is a sufficient statistic for θ, no information is lost by considering instead the density $g(T(x),n|\theta)$. Note that this density considers all possible values of the sufficient statistic which are possible at the termination of the sequential test. The density $g(t(x),n|\theta)$, is obtained via the direct method. The posterior is then computed as

$$P(\theta|T(x),n)= \frac{g(T(x),n|\theta)p(\theta)}{\int_{\theta} g(T(x),n|\theta)p(\theta)d\theta} \tag{6.2}$$

where the denominator is a normalizing constant used to make the posterior integrate to one. In a Bayesian sense, this posterior is sufficient in that it contains all of the prior information and all of the information obtained from the sequential sample.

Using the posterior density, one can compute Bayesian confidence intervals for the parameter θ. For example, $(\underset{\sim}{\theta},\tilde{\theta})$ where $\underset{\sim}{\theta}$ and $\tilde{\theta}$ are determined by

$$\begin{aligned} \int_{-\infty}^{\underset{\sim}{\theta}} P(\theta|T(x),n)d\theta &= \gamma/2 \\ \int_{\tilde{\theta}}^{\infty} P(\theta|T(x),n)d\theta &= \gamma/2 \end{aligned} \tag{6.3}$$

gives a $100(1-\gamma)\%$ Bayesian confidence interval for the state of nature θ. An interval of shortest length can be found by finding $\underset{\sim}{\theta}$ and $\tilde{\theta}$ such that

$$\int_{\underset{\sim}{\theta}}^{\tilde{\theta}} P_n(\theta|T(x),n)d\theta = (1-\gamma) \tag{6.4}$$

and $\underset{\sim}{\theta}-\tilde{\theta}$ is a minimum. The expected value of the posterior

$$E(\theta) = \int_{-\infty}^{\infty} \theta P_n(\theta|T(x),n)d\theta \tag{6.5}$$

can be used as a point estimator for θ. The mode or median of the posterior distribution can also be used for this purpose. Other percentiles of the posterior distribution might also be of interest.

Schmee (1974) gives some discussion on the choice of a prior distribution (often a point of controversy). Box and Tiao (1973)

also treat this problem at some length. Here, the uniform prior is used, although another, more appropriate prior could be specified by the user of these procedures if desired.

6.3 INTERPRETATION OF THE POSTERIOR DISTRIBUTION

In order to interpret the posterior distribution given in (6.2), one should take a Bayesian view of the situation. To do this, consider D, the state of nature, to be a random variable. It is also necessary to specify a prior distribution for D. This can be (and often is) a uniform distribution covering all possible values for the state of nature. Such a prior distribution is called a "non-informative" prior. When a Bayesian approach is used, the sample likelihood and the prior distribution are combined by using Bayes' formula to obtain a posterior distribution expressing all of the available information about the state of nature.

Some statisticians would object to this approach, claiming that the parameter to be estimated is a fixed value and that only the sample from the population is subject to random fluctuation. One who follows the Bayesian approach, however, argues that with the information from the sample and the prior distribution which expresses the prior beliefs about the parameter in question, a posterior distribution can be found which contains all of the available information about the state of nature. While arguments still persist as to which of these approaches is the proper one, the latter one will be used here for purposes of

estimation. Some justification for taking this approach follows.

The best justification for considering the unknown parameter, say D, a random variable, is that in many cases, it is in fact random in a sense. Consider, for example, a situation where lots of 100 transistors are to be accepted or rejected depending on an indication from a sequential sample about how many defectives are in the lot. Even if one is unwilling to consider the number of defectives in a given lot a true random variable, the posterior distribution can still be used to express the relative "degree of belief" for different values of the parameter.

Besides the justification given above, the Bayesian procedure allows one to easily obtain both point and interval estimates for any parameter. In addition, the posterior distribution itself is available and because this is a likelihood procedure and the distribution sample space is countably finite, our estimates depend only on the observed data (and the assumed prior) and are independent of any stopping rules.

The proponents of the classical approach to estimation do not argue against using Bayes' theorem per se, but rather against considering the unknown parameter a random variable, and with the specification of a prior distribution based on subjective probability. The effect that the prior distribution has on the posterior, however, can be minimized by using the "non-informative" prior discussed in the last section.

6.4 ESTIMATION WITH THE HYPERGEOMETRIC DISTRIBUTION

This section treats sequential estimation when sampling is from a finite population. The estimation is to be performed after the completion of a sequential test of a hypothesis, as explained in Section 6.2. Johnson and Kotz (1969) mention some fixed size sample methods for this distribution. Also, a method of finding confidence intervals when sampling from a finite population is given by Katz (1971). Some of these methods could be applied to the situation here; however, the Bayesian approach, as explained above, is used instead.

The following explains how the general method outlined in Section 6.2 is applied to sequential tests of the hypergeometric distribution. The probability mass function of the hypergeometric distribution is

$$h(n;N,n,D) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad (6.6)$$

$$= \frac{D!\ (N-D)!\ n!\ (N-n)!}{N!\ (D-x)!\ x!\ (n-x)!\ (N-D-n+x)!}$$

This is the probability of observing x defects in a fixed size test with sample size n. For a sequential test, the probability of observing x defectives at trial n is always less than or equal to the probability obtained in (6.6). This is due to the stopping rules of the test and their effect on the number of "admissible" paths to a point in the (n,x) space.

The probability of reaching the point (n,x) under the sequential test rules is

$$h_s(x;N,n,D)=K(n,x)\ \frac{D!\ (N-D)!\ (N-n)!}{N!\ (D-x)!\ (N-D-n+x)!} \qquad (6.7)$$

where K(n,x) is the number of "admissible" paths to point (n,x), as dictated by the sequential procedure. K(n,x) is in general quite difficult to determine directly; however, the direct method, as explained in Chapter 3, will allow one to easily compute the probabilities in (6.7).

For each decision point (n,x), (i.e., a point at which the test is terminated and a decision is made in favor of one of the hypotheses) it is desired to obtain the posterior distribution for D, the number of defectives in the population. Following the procedure outlined in Section 6.2, a prior distribution for D is assumed. Here, a uniform prior will be used so that

$$P(D)=\frac{1}{(N+1)} \qquad 0\le D\le N \qquad (6.8)$$

Of course, any desired prior could be substituted for this. Using (6.2) and (6.7),

$$P(D\ x,n)=\frac{h_s(x;N,n,D)P(D)}{\sum_j h_s(x;N,n,j)P(j)} \qquad (6.9)$$

$$=\frac{f(x,N,n,D)}{\sum_j f(x,N,n,j)}$$

where $$f(x,N,n,D)=\frac{(N-D)!}{(D-x)!\ (N-D-n+x)!}$$

and x is the number of defectives observed at trial n.

The denominator in (6.9) is a normalizing constant used to force the posterior to sum to one. The posterior in (6.9) is

independent of any stopping rules and is easily evaluated directly. This gives the desired posterior probability mass function. As explained in Section 6.2, this distribution is interesting in itself, but it is also useful to find quantities such as the expected value or the mode of the distribution and confidence intervals for the number of defectives in the population. This is done as follows.

The expected value (or mean) of the posterior distribution is found by using (6.9) and changing the integration in (6.5) to a summation.

$$\hat{D} = E(D) = \sum_{D} D \cdot P(D|n,x) \tag{6.10}$$

which can be shown (Zacks, 1971) to reduce to

$$\hat{D} = (x+1)(N+2)/(n+2)-1 \tag{6.11}$$

It should be pointed out that the estimation procedure described here is equally valid for the k>2 decision test. The procedure to be used is exactly the same as it is for the two decision test. This is because the estimates depend only on the observed data and not on the particular stopping rules of the test (the stopping rules do, however, dictate points in the (n,x) space at which the test might terminate). In fact, the same procedure is also directly applicable to fixed size procedures.

6.5 NUMERICAL EXAMPLE

Presented here, as an example, are the numerical results of the estimation procedure for Test Plan 1 from Section 4.1. Table 6.1 shows the posterior distribution for D, the expected value of this distribution, confidence limits for D and the actual confidence level. These are given for each decision point on the boundary of the sequential test region.

Table 6.1
Estimates and Confidence Intervals

3 DEFECTIVES AT TRIAL 3 ESTIMATE= 24.688
LOWER LIMIT= 18 UPPER LIMIT= 30
CONFIDENCE LEVEL= 0.9027

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.002
9	10	11	12	13	14	15	16	17
0.003	0.006	0.005	0.007	0.009	0.012	0.014	0.018	0.022
18	19	20	21	22	23	24	25	26
0.026	0.031	0.036	0.042	0.049	0.056	0.064	0.073	0.083
27	28	29	30					
0.093	0.104	0.116	0.129					

3 DEFECTIVES AT TRIAL 4 ESTIMATE= 20.875
LOWER LIMIT= 12 UPPER LIMIT= 28
CONFIDENCE LEVEL= 0.9185

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.001	0.001	0.003	0.005	0.007
9	10	11	12	13	14	15	16	17
0.010	0.014	0.018	0.023	0.029	0.034	0.040	0.046	0.052
18	19	20	21	22	23	24	25	26
0.058	0.063	0.067	0.070	0.073	0.073	0.073	0.068	0.061
27	28	29						
0.052	0.039	0.022						

8 DEFECTIVES AT TRIAL 5 ESTIMATE= 3.571
LOWER LIMIT= 0 UPPER LIMIT= 9
CONFIDENCE LEVEL= 0.9283

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.194	0.161	0.133	0.110	0.090	0.072	0.058	0.046	0.036
9	10	11	12	13	14	15	16	17
0.028	0.021	0.016	0.012	0.008	0.006	0.004	0.003	0.002
18	19	20	21	22	23	24	25	
0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	

4 DEFECTIVES AT TRIAL 6 ESTIMATE= 19.000
LOWER LIMIT= 12 UPPER LIMIT= 26
CONFIDENCE LEVEL= 0.9137

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.003	0.006
9	10	11	12	13	14	15	16	17
0.010	0.015	0.021	0.029	0.037	0.046	0.055	0.063	0.071
18	19	20	21	22	23	24	25	26
0.077	0.081	0.083	0.082	0.078	0.071	0.061	0.048	0.034
27	28							
0.020	0.008							

1 DEFECTIVES AT TRIAL 7 ESTIMATE= 6.111
LOWER LIMIT= 1 UPPER LIMIT= 11
CONFIDENCE LEVEL= 0.9138

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.049	0.096	0.113	0.117	0.112	0.102	0.090	0.076
9	10	11	12	13	14	15	16	17
0.062	0.049	0.038	0.028	0.020	0.014	0.010	0.006	0.004
18	19	20	21	22	23	24		
0.002	0.001	0.001	0.000	0.000	0.000	0.000		

4 DEFECTIVES AT TRIAL 7 ESTIMATE= 16.778
LOWER LIMIT= 10 UPPER LIMIT= 24
CONFIDENCE LEVEL= 0.9298

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.008	0.014
9	10	11	12	13	14	15	16	17
0.021	0.030	0.041	0.051	0.062	0.071	0.079	0.084	0.086
18	19	20	21	22	23	24	25	26
0.085	0.081	0.074	0.064	0.052	0.039	0.027	0.016	0.008
27								
0.002								

4 DEFECTIVES AT TRIAL 8 ESTIMATE= 15.000
LOWER LIMIT= 9 UPPER LIMIT= 22
CONFIDENCE LEVEL= 0.9283

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.015	0.025
9	10	11	12	13	14	15	16	17
0.037	0.050	0.063	0.075	0.084	0.090	0.092	0.090	0.084
18	19	20	21	22	23	24	25	26
0.075	0.063	0.050	0.037	0.025	0.015	0.008	0.003	0.001

2 DEFECTIVES AT TRIAL 10 ESTIMATE= 9.000
LOWER LIMIT= 2 UPPER LIMIT= 11
CONFIDENCE LEVEL= 0.9160

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.037	0.079	0.111	0.128	0.130	0.122	0.106
9	10	11	12	13	14	15	16	17
0.087	0.067	0.049	0.034	0.022	0.014	0.008	0.004	0.002
18	19	20	21	22				
0.001	0.000	0.000	0.000	0.000				

Table 6.1
(cond.)

5 DEFECTIVES AT TRIAL 10 ESTIMATE= 15,000
LOWER LIMIT= 13 UPPER LIMIT= 21
CONFIDENCE LEVEL= 0,9189

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.001	0.003	0.008	0.017
9	10	11	12	13	14	15	16	17
0.030	0.046	0.063	0.080	0.094	0.103	0.107	0.103	0.094
18	19	20	21	22	23	24	25	
0.080	0.062	0.046	0.030	0.017	0.008	0.003	0.001	

5 DEFECTIVES AT TRIAL 11 ESTIMATE= 13,769
LOWER LIMIT= 9 UPPER LIMIT= 19
CONFIDENCE LEVEL= 0,9068

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.001	0.004	0.010	0.020
9	10	11	12	13	14	15	16	17
0.048	0.069	0.089	0.104	0.113	0.114	0.107	0.093	0.075
18	19	20	21	22	23	24		
0.056	0.038	0.023	0.012	0.005	0.002	0.000		

3 DEFECTIVES AT TRIAL 12 ESTIMATE= 8,143
LOWER LIMIT= 4 UPPER LIMIT= 12
CONFIDENCE LEVEL= 0,9034

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.023	0.061	0.099	0.127	0.139	0.135
9	10	11	12	13	14	15	16	17
0.120	0.098	0.074	0.052	0.034	0.020	0.011	0.005	0.002
18	19	20	21					
0.001	0.000	0.000	0.000					

5 DEFECTIVES AT TRIAL 12 ESTIMATE= 12,716
LOWER LIMIT= 8 UPPER LIMIT= 18
CONFIDENCE LEVEL= 0,9303

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.002	0.010	0.025	0.046
9	10	11	12	13	14	15	16	17
0.071	0.095	0.113	0.122	0.121	0.111	0.094	0.073	0.051
18	19	20	21	22	23			
0.033	0.019	0.009	0.004	0.001	0.000			

4 DEFECTIVES AT TRIAL 13 ESTIMATE= 9,667
LOWER LIMIT= 6 UPPER LIMIT= 14
CONFIDENCE LEVEL= 0,9082

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.012	0.039	0.074	0.108	0.131
9	10	11	12	13	14	15	16	17
0.140	0.138	0.115	0.091	0.066	0.043	0.026	0.014	0.006
18	19	20	21					
0.003	0.001	0.000	0.000					

5 DEFECTIVES AT TRIAL 13 ESTIMATE= 11,888
LOWER LIMIT= 7 UPPER LIMIT= 16
CONFIDENCE LEVEL= 0,9226

POSTERIOR DISTRIBUTION OF D

0	1	2	3	4	5	6	7	8
0.000	0.000	0.000	0.000	0.000	0.004	0.017	0.039	0.068
9	10	11	12	13	14	15	16	17
0.097	0.120	0.132	0.131	0.118	0.097	0.078	0.060	0.030
18	19	20	21	22				
0.026	0.007	0.003	0.001	0.000				

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Appendix

Computer Programs Used to Develop and Evaluate the Test Plans

```
1000C
1010C*****************************************************************
1020C*
1030C*        ONE SIDED SEQUENTIAL TEST OF THE HYPERGEOMETRIC DISTRIBUTION
1040C*
1050C*                    WILLIAM Q. MEEKER, JR
1060C*                    INSTITUTE OF ADMINISTRATION AND MANAGEMENT
1070C*                    UNION COLLEGE
1080C*                    SCHENECTADY, NEW YORK 12308
1090C*                    JULY 1974
1100C*
1110C*****************************************************************
1120C
1130         DIMENSION X(200),Y(200)
1140        OPTION LOAD
1150        EQUIVALENCE (A,ALPHA),(B,BETA)
1160        FILENAME INPUT,IOUT
1170        LOGICAL LEV,LDSN,LEST
1180        INTEGER X,Y
1190        INTEGER EV,EST,DSN
1200C        DATA INPUT,IOUT/50,66/
1210 1      PRINT,"INPUT: N,D0,D1,ALPHA,BETA,M0"
1220        INPUT=""
1230        IOUT=""
1240        READ(INPUT,21)XN,D0,D1,ALPHA,BETA,XM0
1250        IF(XN.EQ.0.0) GO TO 99
1260        PRINT,"INPUT: EVAL,DSN,EST"
1270        READ(INPUT,21)EV,DSN,EST
1280 44     FORMAT(10I1)
1290        IOUT=""
1300        INPUT=""
1310        ID0=D0
1320        ID1=D1
1330        N=XN
1340        WRITE(IOUT,15)N,ID0,ID1,ALPHA,BETA
1350 15     FORMAT(///"1N= ",I5/" D0= ",I4/" D1= ",I4/
1360       &  " ALPHA= ",F6.3/" BETA= ",F6.3)
1370 43     FORMAT(6F10.0,3I1)
1380 21     FORMAT(V)
1390        LEV=.FALSE.
1400        LDSN=.FALSE.
1410        LEST=.FALSE.
1420        IF(EV.EQ.1)LEV=.TRUE.
1430        IF(DSN.EQ.1)LDSN=.TRUE.
1440        IF(EST.EQ.1)LEST=.TRUE.
1450        IF(LEV)PRINT 11
```

```
1460 11    FORMAT(" EVALUATION OF REGION REQUESTED")
1470       IF(LEV.AND.LDSN)PRINT 12
1480 12    FORMAT(" DISTRIBUTION OF THE DSN REQUESTED")
1490       IF(LEV.AND.LEST)PRINT 14
1500 14    FORMAT(" ESTIMATION OF PARAMETER REQUESTED")
1510       PRINT,"INPUT: DL,DU,INC"
1520       READ(INPUT,21)DL,DU,DI
1530       IF(DI.EQ.0.0) GO TO 98
1540       IF(DL.GT.DU) GO TO 98
1550       IDI=DI
1560       IDU=DU+1.
1570       IDL=DL+1.
1580       M0=XM0
1590C
1600C         GO TO FIND THE SMALLEST FIXED SIZE TEST
1610C
1620       CALL FIXSIZ(N,IDU,ID1,ALPHA,BETA,NEST,IDL,IDU,IDI:
1630C
1640C         M0<0 NATURAL TRUNCATION
1650C         M0=0   FIXED SIZE TEST
1660C         M0>0     TRUNCATION AT M0
1670C
1680       IF(M0)10,20,40
1690 10    M0=N
1700       PRINT 72
1710 72    FORMAT("0TRUNCATE AT THE NATURAL END OF THE TEST")
1720       GO TO 30
1730 20    M0=NEST
1740       PRINT 73
1750 73    FORMAT("0TRUNCATE AT THE FIXED SIZE SAMPLE")
1760       GO TO 30
1770 40    PRINT 74,M0
1780 74    FORMAT("0TRUNCATE AT TRIAL ",I5)
1790 30    CONTINUE
1800C
1810C         FIGURE WALD REGION FOR DESIRED SEQUENTIAL TEST
1820C
1830       CALL REGION(X,Y,A,B,IDU, D1,N,M0)
1840 112   CONTINUE
1850       PRINT,"INPUT: REGION CHANGE"
1860       READ(INPUT,21)XI,XJ,XK
1870 1212  FORMAT(7F10.0)
1880       II=XI
1890       IF(II.LE.0) GO TO 235
1900       JJ=XJ
1910       KK=XK
1920       X(II)=JJ
1930       Y(II)=KK
1940       WRITE(IOUT,789)II,JJ,KK
1950 789   FORMAT(" REGION CHANGE",4I5)
1960       GO TO 112
```

```
1970 235   CONTINUE
1980C
1990C         GO TO EVALUATE REGION AND ESTIMATE PARAMETER IF DESIRED
2000C
2010       IF(LEV)CALL EVAL(X,Y,LEST,N,LDSN,MO,IDL,IDU,IDI)
2020       GO TO 1
2030 98    PRINT 9999
2040 9999  FORMAT(" *** INPUT ERROR ***")
2050 99    PRINT 999
2060 999   FORMAT(" END OF RUN")
2070       STOP
2080       END
2090       SUBROUTINE FIXSIZ(N,ID0,ID1,ALPHA,BETA,NEST,IDL,IDU,IDI)
2100C -------------------------- ----------------------------------------
2110C
2120C         THIS SUBROUTINE DETERMINES THE UNRANDOMIZED FIXED SIZE TEST
2130C         HAVING THE SMALLEST POSSIBLE SAMPLE SIZE WHILE STILL
2140C         MEETING THE SPECIFIED ERROR PROBABILITY LIMITS.
2150C
2160C ---------------------------------------------------------------------
2170C      DATA IOUT/66/
2180       FILENAME IOUT
2190       DATA IBLNK,ISTR/1H ,1H*/
2200       MARK=IBLNK
2210       IOUT=""
2220       FN=FLNG(N)
2230       FIXD0=FLNG(ID0)+FLNG(N-ID0)-FN
2240       FIXD1=FLNG(ID1)+FLNG(N-ID1)-FN
2250       AM=1.-ALPHA
2260C
2270C         FIND FIRST ESTIMATE USING METHOD OF GUENTHER
2280C
2290       NEST=CHEST(N,ID0,ID1,ALPHA,BETA)
2300C
2310C         TRY IT
2320C
2330       CALL TRY(AM,BETA,ID0,ID1,FIXD0,FIXD1,N,M4,NEST,IC,ALPHAP,BETAP)
2340       IF(M4)2640,2640,2840
2350C
2360C         CONDITIONS MET--REDUCE SAMPLE SIZE
2370C
2380 2840  NEST=NEST-1
2390       CALL TRY(AM,BETA,ID0,ID1,FIXD0,FIXD1,N,M4,NEST,IC,ALPHAP,BETAP)
2400       IF(M4)2840,2640,2840
2410C
2420C         CONDITIONS NOT MET--INCREASE SAMPLE SIZE
2430C
2440 2640  NEST=NEST+1
2450       IF(NEST-N)2960,2960,2680
2460C
2470C         METHOD FAILS--USE DEFAULT VALUES
```

```
2480C
2490 2680   MARK=ISTK
2500        IC=ID0
2510        NEST=N-ID1
2520        GO TO 2760
2530 2960   CALL TRY(AM,BETA,ID0,ID1,FIXD0,FIXD1,N,M4,NEST,IC,ALPHAP,BETAP)
2540        IF(M4)2760,2640,2760
2550 2760   ALPHAP=1.-ALPHAP
2560 49     FORMAT(///" THE FIXED SIZE TEST IS AS FOLLOWS:",1X,A1/" SAMPLE SIZ
2570       &E = ",I5/" CRITICAL VALUE = ",I5/" ALPHA* = ",F10.5/
2580       &" BETA* = ",F10.5)
2590        WRITE(IOUT,49)MARK,NEST,IC,ALPHAP,BETAP
2600        ICG=IC+1
2610        WRITE(IOUT,16)
2620 16     FORMAT(//" FIXED SIZE TEST OC FUNCTION:"//)
2630        WRITE(IOUT,689)
2640 689    FORMAT("0    D  ACCEPT H0  ACCEPT H1")
2650C
2660C          CALCULATE THE OC FOR THE FIXED SIZE TEST
2670C
2680        DO 22 I=IDL,IDU,IDI
2690        ID=I-1
2700        S1=0.
2710        DO 33 ICP1=1,ICG
2720        IC1=ICP1-1
2730 33     S1=S1+THYPER (N,NEST,ID,IC1)
2740        S2=1.-S1
2750 22     WRITE(IOUT,56)ID ,S1,S2
2760 56     FORMAT(1X,I5,2F10.6)
2770        RETURN
2780        END
2790        SUBROUTINE REGION(X,Y,A1,R1,ID0,ID1,N,M0)
2800C
2810C ---------------------------------------------------------------------
2820C
2830C          THIS SUBROUTINE FINDS THE REGIONS FOR A ONE SIDED TEST OF THE
2840C          HYPERGEOMETRIC DISTRIBUTION
2850C
2860C ---------------------------------------------------------------------
2870C
2880        LOGICAL XNG,YNG,Q
2890        INTEGER U5
2900        INTEGER X,Y
2910        DIMENSION X(M0),Y(M0)
2920C       DATA IOUT/66/
2930        FILENAME IOUT
2940        IOUT=""
2950        XLU=FLNG(ID1)+FLNG(N-ID1)-FLNG(N-ID0)-FLNG(ID0)
2960        WRITE(IOUT,16)
2970 16     FORMAT(///" THE WALD REGION IS AS FOLLOWS:"/
2980       &  /"  TRIAL  ACCEPT H0  ACCEPT H1")
```

```
2990        A=ALOG((1.-B1)/A1)
3000        B=ALOG(B1/(1.-A1))
3010C
3020C          INCREMENT TRIAL NUMBER
3030C          U5 IS THE UPPER LIMIT(+1) ON THE NUMBER OF DEFECTIVES
3040C          L5 IS THE LOWER LIMIT(+1) ON THE NUMBER OF DEFECTIVES
3050C
3060        DO 22 I=1,M0
3070        L5=(MAX0(0,I-N+ID1-1))+1
3080        IF(I.GT.1)L5=MAX0(L5,X(I-1))
3090        U5=I+2
3100         Q=.FALSE.
3110        DO 33 K=L5,U5
3120        K2=K-1
3130        IF(K2.GE.I-N+ID  1) GO TO 2380
3140        XL=-1.E35
3150        GO TO 2540
3160 2380   IF(K2.LE.ID0  ) GO TO 2440
3170        XL=1.E35
3180        GO TO 2540
3190C
3200C          FIGURE LIKLIHOOD RATIO AND PROPER ACTION
3210C
3220 2440   XL=XL0+FLNG(N-ID0-I+K2)+FLNG(ID0-K2)-FLNG(ID1-K2)-FLNG(N-ID1-I+K2)
3230 2540   IF(XL.LE.B.OR.K.NE.L5) GO TO 2720
3240        X(I)=-1
3250          Q=.TRUE.
3260 2720   IF(Q) GO TO 2800
3270        IF(XL.LE.B) GO TO 2800
3280        X(I)=K2-1
3290         Q=.TRUE.
3300 2800   IF(XL.LT.A) GO TO 33
3310        IF(K2.GT.  I) GO TO 2900
3320        Y(I)=K2
3330        GO TO 2980
3340 33     CONTINUE
3350 2900    Y(I)=-1
3360C
3370C          PRINT REGION BOUNDRY POINTS
3380C
3390 2980   XNG=X(I).EQ.-1
3400        YNG=Y(I).EQ.-1
3410        IF(I-M0)123,321,123
3420 321    X(I)=(X(I)+Y(I))/2.
3430        Y(I)=X(I)+1
3440 123    IF(XNG.AND.YNG) GO TO 10
3450        IF(XNG) GO TO 11
3460        IF(YNG) GO TO 12
3470        WRITE(IOUT,41)I,X(I),Y(I)
3480 41     FORMAT(1X,I4,4X,I4,7X,I4)
3490        GO TO 20
```

```
      WRITE(IOUT,42)I
   42 FORMAT(1X,I4,7X,1H*,10X,1H*)
      GO TO 20
   11 WRITE(IOUT,43)I,Y(I)
      GO TO 20
   12 WRITE(IOUT,44)I,X(I)
   43 FORMAT(1X,I4,7X,1H*,7X,I4)
   44 FORMAT(1X,I4,4X,I4,10X,1H*)
C
C        CHECK FOR NATURAL END OF TEST
C
   20 IF(Y(I).EQ.X(I)+1) GO TO 40
   22 CONTINUE
   40 RETURN
      END
      FUNCTION CHEST(N,ID0,ID1,ALPHA,BETA)
C-----------------------------------------------------------------------
C
C        FIGURE APROX SAMPLE SIZE USING METHOD OF GUENTHER MODIFIED BY
C        MEEKER
C
C-----------------------------------------------------------------------
      XN=N
      D0=ID0
      D1=ID1
      XB=1.-BETA
      C=0.
    1 C=C+1.
      IF(C.GT.D1+1) GO TO 92
      IL=XNVAL(XN,D1,C,XB)+.9999999
      IU=XNVAL(XN,D0,C,ALPHA)
      IF(IL.GT.IU) GO TO 1
   92 CHEST=IL
      RETURN
      END
      SUBROUTINE TRY(AM,BETA,ID0,ID1,FIXD0,FIXD1,M,M4,NEST,IC,S1,S2)
C-----------------------------------------------------------------------
C
C        TEST TO SEE IF NEST IS A LARGE ENOUGH SAMPLE SIZE TO MEET
C        ERROR LIMITS.
C        M4=1 IF CONDITIONS ARE MET, 0 OTHERWISE
C
C-----------------------------------------------------------------------
      FC=FLNG(N-NEST)+FLNG(NEST)
      FIXAD0=FC+FIXD0
      FIXAD1=FC+FIXD1
      S1=0.
      S2=0.
      M4=1
      I7=MAX0(0,NEST+ID0-N)+1
      I6=MIN0(ID0,NEST)+1
```

```
      DO 22 I=I7,I6
      IC=I-1
      S2=S2+EXP(FIXAD1-FLNG(ID1-IC)-FLNG(N-ID1-NEST+IC)-FLNG(IC)
     &-FLNG(NEST-IC))
      S1=S1+EXP(FIXAD0-FLNG(ID0-IC)-FLNG(N-ID0-NEST+IC)-FLNG(IC)
     &-FLNG(NEST-IC))
      IF(S2.GT.RETA) GO TO 3500
      IF(S1.GT.AM.AND.S2.LE.RETA) GO TO 3540
22    CONTINUE
3500  M4=0
      RETURN
3540  IC=I-1
      RETURN
      END
      FUNCTION XNVAL(XN,D,C,PROB)
      XNVAL=CHISQ(1,2.*C+2.,PROB)
      XNVAL=(XNVAL*(2.*XN+1.-D+.5*C)+C*(2.*D-C))/(4.*D-2.*C+XNVAL)
      RETURN
      END
      SUBROUTINE EVAL(X,Y,EST,N,DSN,M0,IDL,IDU,IDI)
C
C-----------------------------------------------------------------------
C
C        THIS SUBROUTINE EVALUATES THE REGION FOR A SEQUENTIAL TEST OF
C        THE HYPERGEOMETRIC DISTRIBUTION (ONE SIDED TEST).
C        ESTIMATION OF THE PARAMETER IS OFFERED AS AN OPTION
C
C-----------------------------------------------------------------------
C
      INTEGER X,Y,D
      REAL N1,N9,N2
      LOGICAL NOW
      LOGICAL DSN,EST
      FILENAME IOUT
      DIMENSION MARK(50)
      DIMENSION IHOLD(50)
      DIMENSION X(M0),Y(M0)
      DIMENSION A(200),B(200)
      DIMENSION PROB(200)
      EQUIVALENCE (NPO'NT,NP)
      EQUIVALENCE(XMEAN,  MEAN(1))
C
C        NPS IS THE MAXIMUM NUMBER OF DECISION POINTS
C        NMAX IS THE MAXIMUM POPULATION SIZE
C
      DATA ISTEP/9/
      IOUT=""
      IF(.NOT.DSN)WRITE(IOUT,432)
C
C        INCREMENT THE DEFECTIVE NUMBER
C
```

```
      DO 22 ID=IDL,IDU,IDI
      D=ID-1
      IF(DSN)WRITE(IOUT,4921)D
4921  FORMAT(///" NUMBER OF DEFECTIVES = ",I5/)
432   FORMAT(54H0  TRUE D   P(H0)        P(H1)        ASN          VSN  )
43    FORMAT(46H0  TRIAL  P(H0)     P(H1)     P(T)      P(C)    )
      IF(DSN)WRITE(IOUT,43)
      A9=0.
      B9=0.
      N9=0.
      V9=0.
      NP=0
      DO 77 I=1,M0
77    A(I)=0.
C
C        INCREMENT THE TRIAL NUMBER
C
      DO 33 I=1,M0
      N1=I
      I3=I+3
      N2=N1*N1
      DO 88 J=1,M0
88    B(J)=0.
      IF(I-1)25,25,26
C
C        FIGURE PROBS FOR FIRST STEP
C
25    B(2)=FLOAT(D)/FLOAT(N)
      B(1)=1.-B(2)
      GO TO 3540
C
C        BOUNDRY POINTS
C
26    LP1=X(I-1)
      LP2=Y(I-1)
      LPC=X(I)
      LP4=Y(I)
      LP9= MAX0(0,LP1)+1
      IF(LP2.EQ.-1) GO TO 28
27    LP8=MIN0(LP2,I )+1
      GO TO 2560
28    LP8=I+1
C
C        MOVE PROBS TO THE NEXT STEP
C
2560  DO 34 J=LP9,LP8
      Q=MAX0(ID-J,0)
      Q1=N-I+1
      S=Q/Q1
      B(J)=B(J)+A(J)*(1.-S)
      B(J+1)=B(J+1)+A(J)*S
```

```
5030 34     CONTINUE
5040        P6=0.
5050        P7=0.
5060        IF(LPC.LE.LP1) GO TO 52
5070        IF(LPC+1)51,52,51
5080 51     LP=MAXO(0,LP1+1)+1
5090C
5100C       ACCEPT POINTS
5110C
5120        LPC1=LPC+1
5130        DO 222 J=LP,LPC1
5140        P6=P6+B(J)
5150 222    B(J)=0.
5160        A9=A9+P6
5170 52     IF(LP4+1)53,54,53
5180 53     L1=LP4+1
5190        IF(LP4.GT.LP2.AND.LP2.GE.1) GO TO 54
5200        L2=L1
5210        IF(LP4-LP2)3180,3181,3181
5220 3180   L2=LP2+1
5230C
5240C          REJECT POINTS
5250C
5260 3181   DO 333 J=L1,L2
5270        P7=P7+B(J)
5280 333    B(J)=0.
5290        R9=R9+P7
5300 54     P8=P7+P6
5310C
5320C          ACCUMULATE EXPECTED VALUES
5330C
5340        N9=N9+P8*N1
5350        V9=V9+P8*N2
5360        IF(.NOT.USN) GO TO 3500
5370        IF(N9)356,3540,356
5380 356    P10=1.-R9-A9
5390        WRITE(IOUT,445)I,P6,P7,P8,P10
5400 445    FORMAT(1X,I5,4F10.5)
5410 449    FORMAT(3X,2016)
5420 3500   IF(LP1.EQ.-1.OR.LP4.EQ.-1) GO TO 3540
5430C
5440C          CHECK FOR NATURAL END OF TEST
5450C
5460        IF(LP4-LPC.LE.1) GO TO 3580
5470 3540   DO 996 J=1,13
5480 996    A(J)=B(J)
5490 33     CONTINUE
5500 3580   V9=V9-A9*N9
5510        H0=I
5520 229    CONTINUE
5530        IF(.NOT.USN) GO TO 3680
```

```
5540         WRITE(IOUT,432)
5550 3680    WRITE(IOUT,401) D,A9,R9,N9,V9
5560 401     FORMAT(1X,I6,2F12.6,2F12.4)
5570 22      CONTINUE
5580         IF(.NOT.FST) GO TO 99
5590         DO 216 NT=2,M0
5600         KTO=1
5610         IF(X(NT).LE.X(NT-1)) GO TO 4212
5620         ND=X(NT)
5630         GO TO 4213
5640 4212    IF(Y(NT).EQ.-1) GO TO 216
5650         IF(Y(NT).GT.Y(NT-1).AND.Y(NT-1).NE.-1) GO TO 216
5660         ND=Y(NT)
5670         KTO=2
5680C
5690C         FIND CONFIDENCE INTERVAL
5700C
5710 4213    CALL CI(XM,IDL,IDU,IL,IU,PROB,PP,NT,ND,N)
5720         WRITE(IOUT,16)ND,NT,XM,IL,IU,PP
5730 16      FORMAT(//1X,I4," DEFECTIVES AT TRIAL ",I4,
5740        &                "  ESTIMATE= ",F7.3/"   LOWER LIM
5750        &IT= ",I4,"   UPPER LIMIT= ",I4/"   CONFIDENCE LEVEL= ",F6.4)
5760         WRITE(IOUT,17)
5770 17      FORMAT( /" POSTERIOR DISTRIBUTION OF D"/)
5780         IS=1-ISTEP
5790 1       IS=IS+ISTEP
5800         IE=IS+ISTEP-1
5810         IE=MIN0(IE,IDU)
5820         DO 426 IK=IS,IE
5830 426     IHOLD(IK)=IK-1
5840         WRITE(IOUT,449)(IHOLD(I),I=IS,IE)
5850         WRITE(IOUT,215)(PROB(I),I=IS,IE)
5860 215     FORMAT(6X,20F6.3)
5870         WRITE(IOUT,1316)
5880 1316    FORMAT(1H )
5890         IF(IE.NE.IDU) GO TO 1
5900         WRITE(IOUT,1319)
5910 1319    FORMAT(1X,80(1H-))
5920         IF(KTO.EQ.1) GO TO 4212
5930 216     CONTINUE
5940 99      RETURN
5950         END
5960C-----------------------------------------------------------------
5970C
5980C          THIS SUBROUTINE FINDS THE SMALLEST CONFIDENCE INTERVAL
5990C          FOR THE TRUE NUMBER OF DEFECTIVES
6000C
6010C-----------------------------------------------------------------
6020         SUBROUTINE CI(XMEAN,IDL,IDU,IL,IU,PROB,P,NT,ND,N)
6030         DIMENSION PROB(200)
6040         DATA CIL/.1/
```

```
      P=0.
      XMEAN=0.
      IDU=N-NT+ND+1
      IDL=ND+1
      DO 999 I=1,IDL
999   PROB(I)=0.0
      PSUM=0.0
      DO 77 I=IDL,IDU
      J=I-1
      PROB(I)=EXP(FLNG(J)+FLNG(N-J)-FLNG(J-ND)-FLNG(N-J-NT+ND))
77    PSUM=PSUM+PROB(I)
      DO 88 I=IDL,IDU
88    PROB(I)=PROB(I)/PSUM
      DO 22 I=IDL,IDU
22    XMEAN=XMEAN+PROB(I)*FLOAT(I-1)
      IU=IDU
      IL=1
1     IF(PROB(IL).LT.PROB(IU)) GO TO 2
      PH=P+PROB(IU)
      IF(PH.GT.CIL) GO TO 99
      P=PH
      IU=IU-1
      GO TO 1
2     PH=P+PROB(IL)
      IF(PH.GT.CIL) GO TO 99
      P=PH
      IL=IL+1
      GO TO 1
99    P=1.-P
      IU=IU-1
      IL=IL-1
      RETURN
      END
C
C        FUNCTION TO RETURN THE NATURAL LOG FACTORIAL
      FUNCTION FLNG(J)
      DIMENSION F(105)
      DATA MARK/1/
      IF(MARK)20,20,21
20    FLNG=F(J+1)
      RETURN
21    F(1)=0.
      F(2)=0.
      DO 22 I=3,103
      F(I)=F(I-1)+ALOG(FLOAT(I-1))
22    CONTINUE
      MARK=0
      GO TO 20
      END
C
```

```
1000  REM
1010  REM*****************************************************************
1020  REM*
1030  REM*        THIS PROGRAM WILL EVALUATE A THREE DECISION
1040  REM*   SEQUENTIAL TEST REGION. THE REGION MAY BE TRUNCATED.
1050  REM*   THE  TEST REGION IS TO BE READ FROM FILE
1060  REM*   REGON>. THE ASN AND OC FUNCTIONS ARE PROVIDED. THE
1070  REM*   DISTRIBUTION OF THE DECISIVE SAMPLE NUMBER MAY BE
1080  REM*   PRINTED AS AN OPTION.
1090  REM*
1100  REM*
1110  REM*                    WILLIAM O. MEEKER, JR.
1120  REM*                    INSTITUTE OF ADMINISTRATION
1130  REM*                          AND MANAGEMENT
1140  REM*                    UNION COLLEGE
1150  REM*                    SCHENECTADY, NEW YORK 12308
1160  REM*                    JULY 1974
1170  REM*
1180  REM*
1190  REM*****************************************************************
1200  REM
1230  DEF FNZ(X)=INT(X*100000.+.5)/100000.
1240  DIM A$[3]
1250  PRINT "CHARACTERISTICS OF A GIVEN HYP..GEOMETRIC TEST REGION"
1260  PRINT "TWO SIDED TEST"
1270  READ D0,D1,D5,N
1280  PRINT
1290  FILES REGON>
1300  DIM Z[101,4]
1310  READ #1,1
1320  REM-----------------------------------------
1330  REM         READ REGION FROM FILE #1
1340  REM-----------------------------------------
1350  MAT  READ #1;Z
1360  LET T7=1
1370  FOR I=1 TO 100
1380  LET X[I]=Z[I,1]
1390  LET Y[I]=Z[I,2]
1400  LET U[I]=Z[I,3]
1410  LET V[I]=Z[I,4]
1420  NEXT I
1430  LET M1=Z[101,1]
1440  LET M2=Z[101,3]
1450  LET M0=M1 MAX M2
```

```
1460  PRINT "N=";N
1470  PRINT
1480  LET O1=0
1490  PRINT "DO YOU WANT TO SEE THE DSN";
1500  INPUT A$
1510  IF A$ <> "YES" THEN 1720
1520  LET O1=1
1530  GOTO 1840
1540  PRINT
1550  PRINT
1560  PRINT "TRUE";TAB(13);"PROB";TAB(26);"PROB";
1570  PRINT TAB(38);"PROB"
1580  PRINT " D";TAB(11);"ACCEPT H1";TAB(24);"ACCEPT H0";
1590  PRINT TAB(36);"ACCEPT H2";TAB(53);"ASN";TAB(65);"VSN"
1600  REM-----------------------------------------------
1610  REM         SPECIFY TRUE DEFECTIVES
1620  REM-----------------------------------------------
1630  FOR D=D0 TO D1 STEP D5
1640  LET A9=R9=B9=V9=N9=0
1650  MAT A=ZER
1660  IF O1=0 THEN 2140
1670  PRINT
1680  PRINT
1690  PRINT "NUMBER OF DEFECTIVES=";D
1700  PRINT "TRIAL";TAB(13);"PROB";TAB(26);"PROB";
1710  PRINT TAB(38);"PROB";TAB(52);"TOTAL";TAB(64);"PROB"
1720  PRINT "NUMBER";TAB(11);"ACCEPT H1";TAB(24);"ACCEPT H0";
1730  PRINT TAB(36);"ACCEPT H2";TAB(52);"PROB";
1740  PRINT TAB(62);"CONTINUE"
1750  REM-----------------------------------------------
1760  REM           SPECIFY TRIAL NUMBER
1770  REM-----------------------------------------------
1780  FOR N1=1 TO M0
1790  LET N2=N1+N1
1800  MAT B=ZER
1910  IF N1>1 THEN 2280
1820  LET B[2]=D/N
1830  LET B[1]=1-B[2]
1840  GOTO 3880
1850 LET P1=X[N1-1]
1860 LET P4=V[N1-1]
1870 LET P9=0 MAX P1
1880 IF P1=-1 THEN 2400
1890 LET P8=P4 MIN N1
1900 GOTO 2480
1910 LET P8=N1
1920 REM-----------------------------------
1930  REM         FIGURE PROBABILITIES AT THE NEXT STEP
1940  REM-----------------------------------
1950  FOR J=P9 TO P8
1960  LET S=(0 MAX (D-J))/(N-N1+1)
```

```
1970  LET B[J+1]=B[J+1]+A[J+1]*(1-S)
1980  LET B[J+2]=B[J+2]+A[J+1]*S
1990  NEXT J
2000  REM-----------------------------------------
2010  REM         ALLOW PRINTING OF INDIVIDUAL PROBABILITIES
2020  REM-----------------------------------------
2030  LET A$="N"
2040  IF A$ <> "Y" THEN 2820
2050  FOR I=0 TO N1+2
2060  IF B[I+1]=0 THEN 2740
2070  PRINT N1;I;B[I+1]
2080  NEXT I
2090  REM-----------------------------------------------------------------
2100  REM         FIGURE PROBABILITIES OF TERMINATION
2110  REM-----------------------------------------------------------------
2120  LET P3=U[N1]
2130  LET P4=V[N1]
2140  LET P1=X[N1]
2150  LET P2=Y[N1]
2160  LET P5=P6=P8=0
2170  IF P3<P2 OR P3=-1 OR P2=-1 THEN 3140
2180  REM-----------------------------------------
2190  REM       FIGURE PROBABILITIES FOR ACCEPT
2200  REM-----------------------------------------
2210  FOR J=P2 TO P3
2220  IF T7=1 THEN 3060
2230  PRINT "Y";J;B[J+1]
2240  LET P6=P6+B[J+1]
2250  LET B[J+1]=0
2260  NEXT J
2270  LET B9=B9+P6
2280  IF P1=-1 THEN 3360
2290  REM-----------------------------------------
2300  REM      FIGURE PROBABILITIES FOR REJL
2310  REM-----------------------------------------
2320  FOR J=(0 MAX X[N1-1]) TO P1
2330  IF T7=1 THEN 3280
2340  PRINT "X";J;B[J+1]
2350  LET P5=P5+B[J+1]
2360  LET B[J+1]=0
2370  NEXT J
2380  LET A9=A9+P5
2390  IF P4=-1 THEN 3640
2400  LET L1=L2=P4
2410  IF P4 >= Y[N1-1] THEN 3500
2420  LET L2=Y[N1-1]
2430  REM-----------------------------------------
2440  REM    FIGURE PROBABILITIES FOR REJU
2450  REM-----------------------------------------
2460  FOR J=L1 TO L2
2470  IF T7=1 THEN 3560
```

```
2480  PRINT "V";J;B(J+1)
2490  LET P8=P8+B(J+1)
2500  LET B(J+1)=0
2510  NEXT J
2520  LET R9=R9+P8
2530  LET P9=P5+P6+P8
2540  LET V9=V9+N2*P9
2550  LET N9=N9+N1*P9
2560  IF Q1=0 THEN 3840
2570  IF N9=0 THEN 3880
2580  REM-----------------------------------------
2590  REM     PRINT DSN IF DESIRED
2600  REM-----------------------------------------
2610  PRINT N1;TAB(11);FNZ(P5);TAB(24);FNZ(P6);TAB(36);FNZ(P8);
2620  PRINT TAB(50);FNZ(P9);TAB(62);FNZ(1-R9-A9-B9)
2630  IF P1=-1 OR P2=-1 OR P3=-1 OR P4=-1 THEN 3880
2640  IF P2-P1 <= 1 AND P4-P3 <= 1 THEN 3920
2650  MAT A=B
2660  NEXT N1
2670  LET V9=V9-N9^2
2680  IF Q1=0 THEN 4080
2690  PRINT
2700  PRINT
2710  PRINT "TRUE";TAB(13);"PROB";TAB(26);"PROB";
2720  PRINT TAB(38);"PPOB"
2730  PRINT " D";TAB(11);"ACCEPT H1";TAB(24);"ACCEPT H0";
2740  PRINT TAB(36);"ACCEPT H2";TAB(53);"ASN";TAB(65);"VSN"
2750  PRINT D;TAB(11);FNZ(A9);TAB(24);FNZ(B9);
2760  PRINT TAB(36);FNZ(R9);TAB(50);FNZ(N9);TAB(62);FNZ(V9)
2770  NEXT D
2780  DATA 20,60,4,100
2790  END
```